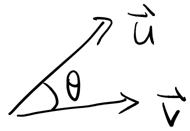
Angle in $\mathbb{R}^2 / \mathbb{R}^3$



$$\vec{u}\cdot\vec{v} = \cos\theta \, \|\vec{u}\| \, \|\vec{v}\|$$

$$\Leftrightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

$$\theta = 90^\circ \Leftrightarrow \vec{u}\cdot\vec{v} = 0$$

Def $x, y \in V$ (inner prod space) are orthogonal (or perpendicular), if $\langle x, y\rangle = 0$.

Def A subset S of is orthogonal if any two distinct vectors are orthogonal.

Ex: ① $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is orthogonal.

② $V = C([-1,1])$ $\quad \langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$

$\{1, x\}$ is orthogonal.

$$\langle 1, x\rangle = \int_{-1}^{1} 1\cdot t\,dt = \int_{-1}^{1} t\,dt = 0$$

$$\|1\| = \sqrt{\langle 1, 1\rangle} = \sqrt{\int_{-1}^{1} 1\cdot 1\cdot dt} = \sqrt{2}.$$

Def If $\|x\| = 1$, then x is a unit vector.

Def S is orthogonal and S contains only unit vectors, S is orthonormal.

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

$S = \{v_1, v_2, \dots\}$ (could be infinite set)

S is orthonormal $\Leftrightarrow \langle v_i, v_j\rangle = \delta_{ij}$

Normalization: $\frac{x}{\|x\|}$ is unit.

Normalization does not affect orthogonality

If $\langle x, y\rangle = 0$, $\langle \frac{x}{\|x\|}, y\rangle = \frac{1}{\|x\|}\langle x, y\rangle = 0$.

If $\langle y, x\rangle = 0$, $\langle y, \frac{x}{\|x\|}\rangle = \frac{1}{\|x\|}\langle y, x\rangle = 0$.

Ex: $V = \mathbb{R}^3$, $\left\{ \underset{v_1}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\-1\\1\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}-1\\1\\2\end{pmatrix}} \right\}$ is orthogonal but not orthonormal.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$\{u_1, u_2, u_3\}$ is orthonormal.

Ex: $H = \{$continuous complex-valued funcs on $[0, 2\pi]\}$

$$\langle f(x), g(x)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\, dt$$

$S = \{e^{int} : n \text{ is an integer}\} \subseteq H$

$$e^{int} = \cos(nt) + i\sin(nt)$$

$$\overline{e^{int}} = \cos(nt) - i\sin(nt) = e^{-int}$$

$m \neq n$, $\langle e^{int}, e^{imt}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{int}\,\overline{e^{imt}}\, dt$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}\, dt$$

$$= \frac{1}{2\pi i(n-m)} e^{i(n-m)t}\Big|_0^{2\pi} = 0.$$

$$\langle e^{int}, e^{int} \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{int}\overline{e^{int}}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{int} e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1.$$

$\Rightarrow \langle e^{imt}, e^{int} \rangle = \delta_{mn}$

$\Rightarrow S$ is orthonormal.

Def S is orthonormal and S is also an ordered basis then S is orthonormal basis.

Ex: ① $\{e_1, e_2, e_3\}$ is orthonormal basis of $\mathbb{R}^3$

② $\left\{ \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$ — — — — of $\mathbb{R}^2$.

Theorem 6-3 $S = \{v_1, \dots, v_k\}$ is orthogonal without $\vec{0}$.

$$y \in \operatorname{Span}(S) \Rightarrow y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Proof: $y \in \operatorname{Span}(S) \Rightarrow y = \sum_{i=1}^{k} a_i v_i$, $a_i \in F$.

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle$$

$$= \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle$$

orthogonal $\Rightarrow \langle v_i, v_j \rangle = 0$ if $i \neq j$

$$= a_j \langle v_j, v_j \rangle$$

$$\Rightarrow a_j = \frac{\langle y, v_j \rangle}{\langle v_j, v_j \rangle} = \frac{\langle y, v_j \rangle}{\|v_j\|^2}.$$

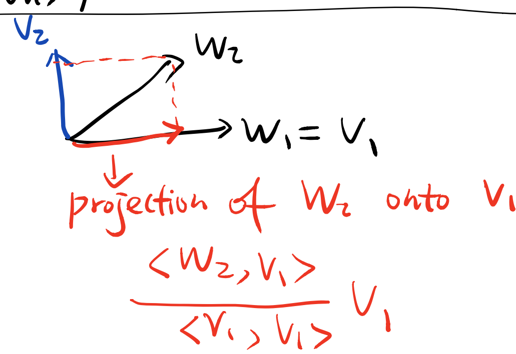
Corollary : $S = \{v_1, \cdots, v_k\}$ is orthonormal

$$y \in \text{Span}(S) \Rightarrow y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

<u>Theorem 6.5</u> If $\beta = \{v_1, \cdots, v_n\}$ is orthonormal basis,

$$[x]_\beta = \begin{pmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_n \rangle \end{pmatrix}$$

Two vectors in $\mathbb{R}^2$ $\{w_1, w_2\}$



$\{v_1, v_2\}$

$$\begin{cases} v_1 = w_1 \\ v_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \end{cases}$$

$$\begin{aligned} &\langle v_2, v_1 \rangle \\ &= \left\langle w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1, v_1 \right\rangle \\ &= \langle w_2, v_1 \rangle - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \langle v_1, v_1 \rangle \\ &= \langle w_2, v_1 \rangle - \langle w_2, v_1 \rangle = 0 \end{aligned}$$

Gram-Schmit Process :

V is inner prod space

$S = \{w_1, \cdots, w_n\}$ is linearly independent

① $\{v_1, \cdots, v_n\}$ is orthogonal.

② $u_i = \frac{v_i}{\|v_i\|}$, $\{u_1, \cdots, u_n\}$ is orthonormal.

$$\begin{cases} v_1 = w_1 \\ v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, \quad 2 \le k \le n \end{cases}$$

$$v_1 = w_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$v_4 = w_4 - \frac{\langle w_4, v_1 \rangle}{\|v_1\|^2} v_1 - \underbrace{\frac{\langle w_4, v_2 \rangle}{\|v_2\|^2}}_{\text{projection of } w_4 \text{ onto } v_2.} v_2 - \frac{\langle w_4, v_3 \rangle}{\|v_3\|^2} v_3$$

$$\langle v_3, v_1 \rangle = \left\langle w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2, \; v_1 \right\rangle$$

$$= \langle w_3, v_1 \rangle - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} \underset{0}{\underset{\|}{\langle v_2, v_1 \rangle}}$$

$$= \langle w_3, v_1 \rangle - \langle w_3, v_1 \rangle$$

$$= 0.$$

Ex: $V = P_2(\mathbb{R})$ $\quad \langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt.$

$$S = \{\underset{w_1}{1}, \underset{w_2}{x}, \underset{w_3}{x^2}\}$$

① Apply Gram-Schmit:

$$v_1 = w_1 = 1, \quad \|v_1\|^2 = \int_{-1}^{1} 1 \cdot 1 \cdot dt = 2$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= x - \frac{\int_{-1}^{1} t \cdot 1 \cdot dt}{2} 1 = x.$$

$$\|v_2\|^2 = \langle v_2, v_2 \rangle = \int_{-1}^{1} t^2 dt = \frac{2}{3}.$$

$$V_3 = W_3 - \frac{\langle W_3, V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle W_3, V_2\rangle}{\|V_2\|^2} V_2$$

$$= x^2 - \frac{\int_{-1}^{1} t^2 \cdot 1 \cdot dt}{2} 1 - \frac{\int_{-1}^{1} t^2 \cdot t \, dt}{2/3} x$$

$$= x^2 - \frac{1}{3}.$$

$\{1, x, x^2 - \frac{1}{3}\}$ is orthogonal.

$$\|V_3\|^2 = \int_{-1}^{1} (t^2 - \tfrac{1}{3})^2 dt = \int_{-1}^{1} (t^4 - \tfrac{2}{3}t^2 + \tfrac{1}{9}) dt$$

$$= \frac{8}{45}$$

② $u_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$$u_2 = \frac{V_2}{\|V_2\|} = \frac{x}{\sqrt{2/3}} = \sqrt{\tfrac{3}{2}}\, x$$

$$u_3 = \frac{V_3}{\|V_3\|} = \sqrt{\tfrac{5}{8}}\,(3x^2 - 1).$$

$\beta = \{u_1, u_2, u_3\}$ is orthonormal basis.

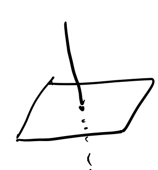
$f(x) = 1 + 2x + 3x^2$

$$[f]_\beta = \begin{pmatrix} \langle f, u_1\rangle \\ \langle f, u_2\rangle \\ \langle f, u_3\rangle \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} (1+2t+3t^2)\frac{\sqrt{2}}{2}\, dt \\ \int_{-1}^{1} (1+2t+3t^2)\sqrt{\tfrac{3}{2}}\, t\, dt \\ \int_{-1}^{1} (1+2t+3t^2)\sqrt{\tfrac{5}{8}}(3t^2-1)\, dt \end{pmatrix}.$$

Def $S \subseteq V$ is a subspace

$S^\perp$ ("S perp") is defined as

$\{v \in V : \langle v, w\rangle = 0,\ \forall w \in S\}$

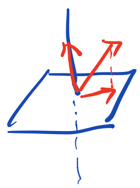


Ex: $V = \mathbb{R}^3$, $S = \{x\text{-}y \text{ plane}\}$ $S^\perp = \{z\text{-axis}\}$

$V^\perp = \{\vec{0}\}$, $\{\vec{0}\}^\perp = V$.

$\text{Span}\{1, x\}^\perp = \text{Span}\{x^2 - \frac{1}{3}\}$

Theorem 6.6 W is a finite-dim subspace of an inner prod space of V



① $\Rightarrow \forall y \in U, \exists u \in W, \exists z \in W^\perp$ s.t.

$$y = u + z$$

② If W has an orthonormal basis $\{v_1, \cdots, v_k\}$

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Proof: Let $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i \in W$

Let $z = y - u$, then $z + u = y$.

Only need to show $z \in W^\perp$.

$z \in W^\perp \Leftrightarrow \langle z, v \rangle = 0$, $\forall v \in W$

$\Leftrightarrow \langle z, \sum_{i=1}^{k} a_i v_i \rangle = 0$, $\forall a_i \in F$.

$\Leftrightarrow \sum_{i=1}^{k} \overline{a_i} \langle z, v_i \rangle = 0$, $\forall a_i \in F$

"$\Leftarrow$" trivial
"$\Rightarrow$" Set $\overline{a_i} = 1$ and others are 0, $\overline{a_i} \langle z, v_i \rangle = \langle z, v_i \rangle$

$\Leftrightarrow$ $\langle z, v_i \rangle = 0$, $\forall i$

Need to verify $\langle z, v_i \rangle = 0$, $\forall i$

$\langle z, v_i \rangle = \langle y - u, v_i \rangle$

$$= \langle y - \sum_{i=1}^{k} \langle y, v_i \rangle v_i \, , \, v_i \rangle$$

$$= \langle y - \sum_{j=1}^{k} \langle y, v_j \rangle v_j \, , \, v_i \rangle$$

$$= \langle y, v_i \rangle - \sum_{j=1}^{k} \langle y, v_j \rangle \underbrace{\langle v_j, v_i \rangle}_{\delta_{ij}}$$

$$= \langle y, v_i \rangle - \langle y, v_i \rangle$$

$$= 0.$$