Theorem 6.9 V is finite dim, for linear $T: V \to V$
there exists a unique $T^*: V \to V$
s.t. $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, $\forall x, y \in V$.

Def (Adjoint) For a linear $T: V \to V$ where V could be infinite dimensional, if there exists a linear $T^*: V \to V$ s.t.
$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, $\forall x, y \in V$,
T^* is called adjoint of T.

Theorem 6.10 β is an orthonormal basis,

$$[T^*]_\beta = [T]_\beta^* .$$

Example: $V = P_1(\mathbb{R})$, $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$
$T[f] = f'(x)$. Find T^*.

Solution: ① Gram-Schmidt and Normalization
$\Rightarrow \beta = \{\underbrace{1}_{v_1}, \underbrace{2\sqrt{3}x - \sqrt{3}}_{v_2}\}$ is orthonormal basis.

② $[T]_\beta$

$T(v_1) = 0 = 0 \cdot v_1 + 0 \cdot v_2$

$T(v_2) = 2\sqrt{3} = 2\sqrt{3}v_1 + 0 \cdot v_2$

$$[T]_\beta = \begin{bmatrix} 0 & 2\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

③ $[T^*]_\beta = \begin{bmatrix} 0 & 0 \\ 2\sqrt{3} & 0 \end{bmatrix}$

$$T^*(v_1) = 0 \cdot v_1 + 2\sqrt{3} v_2 = 2\sqrt{3}(2\sqrt{3}x - \sqrt{3})$$
$$= 12x - 6$$

$$T^*(v_2) = 0 \cdot v_1 + 0 v_2 = 0$$

$$T^*(av_1 + bv_2) = aT^*(v_1) + bT^*(v_2)$$
$$= a(12x-6)$$

Problem: Assume T^* exists, show it's unique.

Proof: Assume $\langle T(x), y \rangle = \langle x, T_1^*(y) \rangle$

$$\langle T(x), y \rangle = \langle x, T_2^*(y) \rangle, \quad \forall x, y \in V$$

$$\Rightarrow \quad 0 = \langle x, T_1^*(y) \rangle - \langle x, T_2^*(y) \rangle$$

$$= \langle \underline{x}, \underline{\underline{T_1^*(y) - T_2^*(y)}} \rangle$$

Set $x = T_1^*(y) - T_2^*(y)$,

$$\Rightarrow 0 = \langle T_1^*(y) - T_2^*(y), T_1^*(y) - T_2^*(y) \rangle$$
$$= \| T_1^*(y) - T_2^*(y) \|^2$$
$$\Rightarrow T_1^*(y) = T_2^*(y), \quad \forall y.$$

Theorem 6.11 $\quad T$ and $U: V \longrightarrow V$

Assume T^*, U^* exist.

$$\begin{cases} ① & (T+U)^* = T^* + U^* \\ ② & (cT)^* = \bar{c} T^*, \quad c \in F \end{cases}$$

③ $(TU)^* = U^*T^*$

④ $(T^*)^* = T$

⑤ $I^* = I$.

Proof: ① $\langle (T+U)x, y\rangle$

$= \langle T(x), y\rangle + \langle U(x), y\rangle$

$= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle$

$= \langle x, T^*(y) + U^*(y)\rangle$

$= \langle x, \underline{\underline{(T^*+U^*)}}(y)\rangle$

$\Rightarrow (T+U)^* = T^* + U^*$.

② $\langle cT(x), y\rangle$

$= c\langle T(x), y\rangle$

$= c\langle x, T^*(y)\rangle$

$= \langle x, \bar{c}T^*(y)\rangle$

$\Rightarrow (cT)^* = \bar{c}T^*$

③, ④, ⑤ are similar.

Corollary: $A, B \in \mathbb{C}^{n\times n}$

① $(A+B)^* = A^* + B^*$

② $(cA)^* = \bar{c}A^*$

③ $(AB)^* = B^*A^*$

④ $(A^*)^* = A$

⑤ $(I)^* = I$

Proof: $T = L_A$

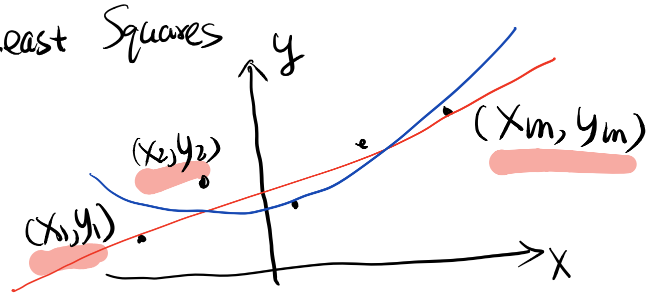
$U = L_B$

Corollary: $\langle x, T(y)\rangle = \langle T^*(x), y\rangle, \ \forall x, y \in F.$

Proof: $\langle T^*(x), y\rangle = \langle x, (T^*)^*(y)\rangle$

$= \langle x, T(y)\rangle.$

Least Squares



Best line fit
quadractic

Line fit: $y = cx + d$ $\quad A \ x = b$

over-determined
$$\begin{cases} y_1 = cx_1 + d \\ y_2 = cx_2 + d \\ \vdots \\ y_m = cx_m + d \end{cases} \Rightarrow \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Fitting Error $= |cx_1 + d - y_1|^2 + |cx_2 + d - y_2|^2 + \cdots + |cx_m + d - y_m|^2$
$\qquad \| $
$\|Ax - b\|^2$

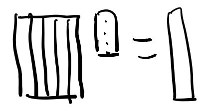
Quadratic fit: $y = bx^2 + cx + d$

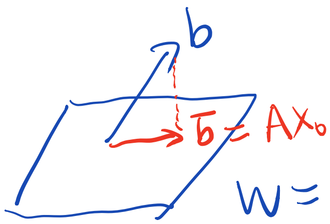
over-determined
$$\begin{cases} y_1 = bx_1^2 + cx_1 + d \\ \vdots \\ y_m = bx_m^2 + cx_m + d \end{cases} \Rightarrow \begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_m^2 & x_m & 1 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$
$$A \quad x \quad = b$$

Consider $\underset{n}{A} x = b$ $\quad A \in F^{m\times n}, \ x \in F^{n\times 1}, \ y \in F^{m\times 1}$

$m > n$ $\quad m \boxed{A} \boxed{x} = \boxed{b}$

If $\hat{x}$ is a sol, then $A\hat{x}=b \Rightarrow b \in Col(A)$





$b \in F^m$

$W = Col(A)$ $\quad$ $\dim(W) = rank(A) \leq n$

$\bar{b}$ is the projection of b onto W.

Then $Ax = \bar{b}$ has solutions, Let x_0 be one sol,

then $Ax_0 = \bar{b}$

Claim $\|Ax_0 - b\| \leq \|Ax - b\|$, $\forall x \in F^n$

Proof: $\|Ax_0 - b\| = \|\bar{b} - b\| \leq \|b - y\|$, $\forall y \in Col(A)$

$\forall x \in F^n$, $Ax \in Col(A) \Rightarrow \|Ax_0 - b\| \leq \|b - Ax\|$

Lemma 1: $A \in F^{m\times n}$, $x \in F^n$, $y \in F^m$

$$\langle Ax, y\rangle_{F^m} = \langle x, A^* y\rangle_{F^n}$$

Proof: $\langle Ax, y\rangle = y^* Ax = (y^* A)x$

$= (A^* y)^* x$

$= \langle x, A^* y\rangle$.

Lemma 2: $A \in F^{m\times n}$, $rank(A^*A) = rank(A)$

$A^* \in F^{n\times m}$, $A^*A \in F^{n\times n}$

Dimension Theorem

$$T: V \to W$$

$$\dim(\underline{\underline{V}}) = \dim(N(T)) + \dim(R(T))$$

$$N(T) \subseteq V \qquad R(T) \subseteq W$$

Proof:

$$L_{A^*A}: F^n \to F^n$$

$$n = \text{Nullity}(L_{A^*A}) + \text{rank}(A^*A)$$

$$L_A: F^n \to F^m$$

$$n = \text{Nullity}(L_A) + \text{rank}(A)$$

ONLY Need to show $\text{Nullity}(L_{A^*A}) = \text{Nullity}(L_A)$

We want to show $Ax = 0 \Leftrightarrow A^*Ax = 0$.

"$\Rightarrow$" is obvious

"$\Leftarrow$" $0 = \langle A^*Ax, x\rangle = \langle Ax, (A^*)^*x\rangle$

$$= \langle Ax, Ax\rangle$$

$$\Rightarrow \|Ax\|^2 = 0$$

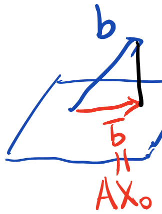
$$\Rightarrow Ax = 0.$$

Assume A has independent cols, then $\dim(\text{Col}(A)) = n$.

$\Rightarrow \text{rank}(A) = n$ $\qquad A^*A \in F^{n\times n}$

$\Rightarrow \text{rank}(A^*A) = \text{rank}(A) = n$

$\Rightarrow A^*A$ is invertible



$W = Col(A)$

If $\bar{b} = Ax_0$ is projection of b on W,

then $\bar{b} - b \in W^{\perp}$ (Theorem 6.6).

$\forall x \in F^n$, $Ax \in Col(A) = W$

$$\Rightarrow 0 = \langle Ax, \bar{b} - b \rangle$$

$$= \langle Ax, Ax_0 - b \rangle$$

$$= \langle x, A^*(Ax_0 - b) \rangle$$

Set $x = A^*(Ax_0 - b) \Rightarrow A^*(Ax_0 - b) = \vec{0}$

$$\Rightarrow (A^*A)x_0 = A^*b$$

$$\Rightarrow x_0 = (A^*A)^{-1}A^*b.$$