- In Chapter 6, $F = \mathbb{R}$ or $\mathbb{C}$
- Foundamental Thm of Algebra: $p(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$

  $a_i \in \mathbb{C} \Rightarrow p(t)$ has $n$ complex roots including repeated ones.
- A matrix $A \in F^{n \times n}$ is diagonalizable $\Leftrightarrow$ ① A has n eigenvalues, ② Alg Mul = Geo Mul

## 6.4: Normal and Self-Adjoint Operators.

**Lemma:** $T: V \to V$, $\dim(V) = n$

$T$ has an eigenvector $\Rightarrow$ So does $T^*$.

**Proof:** Assume $T(v) = \lambda v$, $v \neq \vec{0}$.

$\Rightarrow (T - \lambda I)(v) = \vec{0}$

$\Rightarrow \forall x \in V$, $0 = \langle \vec{0}, x \rangle = \langle (T - \lambda I) v, x \rangle$

$= \langle v, (T - \lambda I)^*(x) \rangle$

$= \langle v, (T^* - \bar{\lambda} I)(x) \rangle$

$\Rightarrow (T^* - \bar{\lambda} I)(x) \neq v$, $\forall x \in V$.

$\Rightarrow U = T^* - \bar{\lambda} I$ is not onto

$U: V \to V$

Dim Theorem $\underset{= n}{\dim(V)} = \underset{< n}{\dim(R(U))} + \underset{\geq 1}{\dim(N(U))}$

$\Rightarrow \dim(N(U)) \geq 1$

$\Rightarrow \exists y \in N(U)$, $y \neq \vec{0}$

$\Rightarrow (T^* - \bar{\lambda} I)(y) = \vec{0} \Rightarrow T^*(y) = \bar{\lambda} y$.

Theorem 6.14 (Schur)

$T: V \to V \quad \dim(V) = n$

$T$ has $n$ eigenvalues including repeated ones

$\Rightarrow$ there exists orthonormal basis $\gamma$ s.t. $[T]_\gamma$ is upper triangular

$$\begin{bmatrix} A_{11} & & * \\ & A_{22} & \\ & & \ddots \\ 0 & & & A_{nn} \end{bmatrix}$$

Pick any basis $\gamma$, $[T]_\gamma$ $\quad p(t) = \det([T]_\gamma - tI)$

Pick another basis $\beta$, $[T]_\beta = Q^{-1}[T]_\gamma Q$.

$$\det([T]_\beta - tI) = \det(Q^{-1}[T]_\gamma Q - tI)$$

$$= \det\left(Q^{-1}[T]_\gamma Q - Q^{-1}(tI)Q\right)$$

$$= \det[Q^{-1}([T]_\gamma - tI)Q]$$

$$= \det[([T]_\gamma - tI)QQ^{-1}].$$

Sketchy Proof: Step I: $A \in F^{n \times n}$

assume $\det(A - tI)$ has $n$ roots.

Want to show $\exists P$ s.t. $P^{-1}AP = U$ is upper triangular.

Math Induction

1) $n=1$, trivial.

2) Assume it's true for $(n-1)\times(n-1)$ matrices

Let $\lambda$ be one eigenvalue of $A$.

$v_1$ ---- eigenvector of $A$.

$$Av_1 = \lambda v_1$$

Extend $\{v_1\}$ to $\{v_1, \dots, v_n\}$ a basis of $F^n$.

$$P = [v_1\ v_2\ \cdots\ v_n]$$

$$AP = [Av_1\ Av_2\ \cdots\ Av_n]$$

$$= [\lambda v_1\ Av_2\ \cdots\ Av_n]$$

$$Pe_1 = v_1 \Rightarrow e_1 = P^{-1} v_1$$

$$P^{-1}AP = [\underbrace{P^{-1}\lambda v_1}_{\lambda(P^{-1}v_1) = \lambda e_1 = \begin{bmatrix}\lambda\\0\\\vdots\\0\end{bmatrix}}\ P^{-1}Av_2\ \cdots\ P^{-1}Av_n]$$

$$= \left[\begin{array}{c|c} \lambda & u \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} \lambda & u \\ \hline 0 & B \end{array}\right]$$

$B$: $(n-1)\times(n-1)$

$P^{-1}AP$ and $A$ have the same characteristic poly

thus same roots.

$$\det(P^{-1}AP - tI) = \det\left(\begin{bmatrix}\lambda & u \\ 0 & B\end{bmatrix} - tI\right)$$

$$= \det \left[\begin{array}{c|c} \lambda - t & u \\ \hline 0 & B - tI \end{array}\right]$$

$$= \det(\lambda - t) \det(B - tI)$$

$$= (\lambda - t) \det(B - tI)$$

$\Rightarrow \det(B - tI)$ has $(n-1)$ roots.

Induction Hypothesis $\Rightarrow \exists Q$ s.t. $Q^{-1}BQ$ is upper triangular

$$R = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array}\right]$$

$$R^{-1} = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & Q^{-1} & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^{-1} \end{array}\right]$$

$$M = PR$$

$$M^{-1}AM = R^{-1} \underbrace{P^{-1}AP}R$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^{-1} \end{array}\right]\left[\begin{array}{c|c} \lambda & u \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right]$$

$$= \left(\begin{array}{c|c} \lambda & uQ \\ \hline 0 & \\ \vdots & Q^{-1}BQ \\ 0 & \end{array}\right)$$

$\hookrightarrow$ upper triangular.

Step II: $\gamma' = \{e_1, \cdots, e_n\}$

$[T]_{\gamma'}$ has $n$ eigenvalues

$\Rightarrow \exists\, Q$ s.t. $Q^{-1}[T]_{\gamma'}Q = U$ is upper triangular

$\Rightarrow \exists$ a basis $\beta'$ s.t. $[T]_{\beta'} = Q^{-1}[T]_{\gamma'}Q = U$

Step III: Gram-Schmidt and Normalization $\Rightarrow \beta$

Check $[T]_\beta$ is still upper triangular.

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**Def** $T: V \to V$ is normal if $TT^* = T^*T$

A matrix $A \in \mathbb{C}^{n\times n}$ is normal $AA^* = A^*A$.

Ex 1: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad AA^* = I = A^*A$.

$$A^* = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

Ex 2: $A^T = -A$ skew-symmetric

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

$A$ is real skew-symmetric $\Rightarrow$ $A$ is normal

$$AA^* = AA^T = A(-A) = -A^2$$
$$A^*A = A^TA = (-A)A = -A^2$$

Ex 3: $T: V \to V$ $V$ has an orthonormal basis $\beta$ s.t. $[T]_\beta = D$ is diagonal.

$\Rightarrow T$ is normal.

$$T: V \to V \quad [T]_\beta = D$$
$$T^*: V \to V \quad [T^*]_\beta = [T]_\beta^* = D^*$$
$$TT^*: V \to V \quad [TT^*]_\beta = [T]_\beta [T^*]_\beta = DD^*$$
$$\parallel$$
$$T^*T: V \to V \quad [T^*T]_\beta = [T^*]_\beta [T]_\beta = D^*D$$

Ex 4: Any real symmetric or complex Hermitian matrix is normal.

$$A^* = A \Rightarrow A^*A = A^2 = AA^*.$$

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Theorem 6.15 $T: V \to V$ is normal.

① $\|T(x)\| = \|T^*(x)\|$, $\forall x \in V$.

② $T - cI$ is normal, $\forall c \in F$

③ $T(x) = \lambda(x) \Rightarrow T^*(x) = \bar{\lambda}(x)$

④ $\left.\begin{array}{l} T(x_1)=\lambda_1 x_1 \\ T(x_2)=\lambda_2 x_2 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Rightarrow \langle x_1, x_2\rangle = 0.$

Proof: ① $\langle T(x), T(x)\rangle = \langle T^*T(x), x\rangle$

$= \langle TT^*(x), x\rangle$

$= \langle T^*(x), T^*(x)\rangle$

② HW

③ $U = T-\lambda I$ , $U(x)=\vec{0}$

② $\Rightarrow$ $U$ is normal.

① $\Rightarrow$ $0 = \|U(x)\| = \|U^*(x)\|$

$\Rightarrow$ $U^*(x) = \vec{0}$

$\Rightarrow$ $(T^*-\bar{\lambda} I)(x) = \vec{0}$

$\Rightarrow$ $T^*(x) = \bar{\lambda} x$.

④ $\lambda_1 \langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle$

$= \langle T(x_1), x_2\rangle$

$= \langle x_1, T^*(x_2)\rangle$

$= \langle x_1, \bar{\lambda}_2 x_2\rangle$

$= \lambda_2 \langle x_1, x_2\rangle$

$\Rightarrow$ $(\lambda_1 - \lambda_2)\langle x_1, x_2\rangle = 0$

$\Rightarrow$ $\langle x_1, x_2\rangle = 0$.

Theorem 6.16 $T: V \to V$ $\dim(V) = n$, $F = \mathbb{C}$

$T$ is normal $\iff$ $\exists$ orthonormal basis of $V$

$T^*T = TT^*$ consisting of eigenvectors of $T$.

Proof: "$\Rightarrow$"

$F = \mathbb{C} \Rightarrow T$ has $n$ eigenvalues including repeated ones.

Schur's Theorem $\Rightarrow$ $\exists$ orthonormal $\beta = \{v_1, \cdots, v_n\}$

s.t. $[T]_\beta = A$ is upper triangular.

Claim $v_i$ are eigenvectors

Proof by induction

1) $v_1$ must be eigenvector:

$$[T]_\beta = \begin{bmatrix} A_{11} & \ddots & * \\ 0 & 0 & \ddots \\ \vdots & & \\ 0 & & A_{nn} \end{bmatrix}$$

$$[T(v_1)]_\beta = [T]_\beta [v_1]_\beta = [T]_\beta e_1 = \begin{bmatrix} A_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow T(v_1) = A_{11} v_1$$

2) Assume $v_1, \cdots, v_{k-1}$ are eigenvectors

Theorem 6.15 $\Rightarrow$ $T^*(v_j) = \bar{\lambda}_j v_j$, $\forall j = 1, 2, \cdots, k-1$.

$$\forall j \le k-1, \quad \underset{A_{jk}}{\underset{\parallel}{\langle T(v_k), v_j \rangle}} = \langle v_k, T^*(v_j) \rangle = \langle v_k, \bar{\lambda}_j v_j \rangle$$

$$= \lambda_j \langle v_k, v_j \rangle = 0$$

$$T(v_k) = \langle T(v_k), v_1 \rangle v_1 + \langle T(v_k), v_2 \rangle v_2 + \cdots + \langle T(v_k), v_n \rangle v_n$$

$$[T]_\beta = A = \begin{array}{c} \begin{bmatrix} \ddots & 0 & & \\ & \vdots & & \\ & 0 & & \\ & A_{kk} & & \\ O & 0 & & \\ & \vdots & \ddots & \\ & 0 & 0 & \end{bmatrix} \\ k\text{-th col} \end{array}$$

$$\Rightarrow [T(V_k)]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad T(V_k) = \underline{\underline{A_{kk}}}\, V_k .$$

" $\Leftarrow$ "

$\beta = \{v_1, \cdots, v_n\}$ are eigenvectors

$$\Rightarrow T(v_i) = \lambda_i v_i$$

$$\Rightarrow [T]_\beta = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} = D$$

$\beta$ is orthonormal

$$\Rightarrow [T^*]_\beta = [T]_\beta^* = \begin{pmatrix} \bar\lambda_1 & & O \\ & \ddots & \\ O & & \bar\lambda_n \end{pmatrix} = \bar D .$$

$$[TT^*]_\beta = [T]_\beta [T^*]_\beta = D\bar D = \bar D D = [T^*]_\beta [T]_\beta = [T^*T]_\beta$$

$$\Rightarrow TT^* = T^*T .$$

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Remark: $[T^*]_\beta = \begin{pmatrix} \bar\lambda_1 & & O \\ & \ddots & \\ O & & \bar\lambda_n \end{pmatrix} \Rightarrow T^*(v_i) = \bar\lambda_i v_i .$