Def $T: V \to V$ is self-adjoint (or Hermitian) if $T^* = T$.

$\langle T(x), y \rangle = \langle x, T(y) \rangle$

$A \in \mathbb{C}^{n\times n}$ is self-adjoint (or Hermitian) if $A^* = A$.

$\langle Ax, y \rangle = \langle x, Ay \rangle$ $A \in F^{n\times n}$, $x, y \in F^n$

Lemma $\dim(V) = n$, $T: V \to V$ is self-adjoint

(a) All eigenvalues are real for either $F = \mathbb{C}$ or $F = \mathbb{R}$.

(b) If $F = \mathbb{R}$, T has n real eigenvalues.

(If $F = \mathbb{C}$, fundamental theorem of algebra

$\Rightarrow$ Any linear operator T has n eigenvalues)

Proof: (a) Assume $T(v) = \lambda v$, $v \neq \vec{0}$.

$T^* = T \Rightarrow T^*T = T^2 = TT^* \Rightarrow T$ is normal.

$\Rightarrow T^*(v) = \bar{\lambda} v$.

$\Rightarrow \lambda v = T(v) = T^*(v) = \bar{\lambda} v$

$\Rightarrow (\underline{\lambda - \bar{\lambda}})\underline{v} = \vec{0}$

$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

(b) $T: V \to V$ over $F = \mathbb{R}$

β is orthonormal basis of V

$A = [T]_\beta \in \mathbb{R}^{n\times n}$, $T^* = T \Rightarrow A^* = A$.

Consider $L_A : \mathbb{C}^n \to \mathbb{C}^n$
$x \mapsto Ax$ $\quad \gamma = \{e_1, \dots, e_n\}$ for $\mathbb{C}^n$

$$\underset{[L_{A^*}]_\gamma}{\underset{\parallel}{A^*}} = \underset{[L_A]_\gamma}{\underset{\parallel}{A}} \Rightarrow \underline{(L_A)^* = L_A}$$

Apply (a) to $L_A \Rightarrow$ All eigenvalues of L_A are real.

Fundamental Thm of Algebra $\Rightarrow L_A$ has n eigenvalues

$\Rightarrow L_A$ has n real eigenvalues

$\Rightarrow$ $\underline{\det(A - tI)}$ has n real roots.

Theorem 6.17 $\quad \dim(V) = n$, $\underline{F = \mathbb{R}}$

T is self-adjoint $\Leftrightarrow$ there exist an orthonormal basis β consisting of eigenvectors of T.

Proof: "$\Leftarrow$" HW

"$\Rightarrow$" $\quad T = T^* \overset{\text{(Lemma)}}{\Rightarrow} T$ has n real eigenvalues.

Schur's Thm $\Rightarrow$ orthonormal basis β s.t.
$A = [T]_\beta$ is upper triangular.

$$\underline{A^*} = [T]_\beta^* = [T^*]_\beta = [T]_\beta = \underline{A}$$

$$\begin{bmatrix} \diagdown & 0 \\ * & \diagdown \end{bmatrix} \qquad\qquad \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$$

$\Rightarrow A$ is diagonal.

$\Rightarrow [T]_\beta = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \Rightarrow T(v_i) = d_i v_i$.

Ex: ① $\begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ are Hermitian

② $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ is $\begin{cases} \text{not Hermitian} \\ \text{real skew-symmetric thus normal.} \end{cases}$

③ $A^* = -A \Rightarrow A^*A = -A^2 = AA^* \Rightarrow A$ is normal.

④ $\begin{bmatrix} i & i \\ i & 1 \end{bmatrix}$ is not normal $AA^* = A^*A$.

6.5

Def $T: V \to V$ is called $\begin{cases} \text{unitary for } F = \mathbb{C} \\ \text{orthogonal for } F = \mathbb{R}. \end{cases}$

if $\begin{cases} \text{① } \dim(V) = n, \quad \|T(x)\| = \|x\|, \ \forall x \in V. \\ \text{② infinite dim, } T \text{ is onto and } \|T(x)\| = \|x\|, \ \forall x \in V. \end{cases}$

(Assume $\|T(x)\| = \|x\|$, if $T(u) = T(v)$, $T(u-v) = \vec{0}$
$0 = \|\vec{0}\| = \|T(u-v)\| = \|u-v\| \Rightarrow u = v$.)

Theorem 6.18 $\dim(V) = n$

The following are equivalent:

(a) $T^*T = I \Leftrightarrow [T^*T]_\beta = [I]_\beta \Leftrightarrow [T^*]_\beta [T]_\beta = I$

(b) $TT^* = I$

(c) $\langle T(x), T(y)\rangle = \langle x, y\rangle, \quad \forall x, y \in V$

(d) If β is an orthonormal basis, then $T(\beta)$ - - - - - - - - -

(e) $\exists$ an orthonormal basis β s.t. $T(\beta)$ is . - - - - - -

(f) $\|T(x)\| = \|x\|, \quad \forall x \in V.$

Proof: (a) $\Leftrightarrow$ (b) by matrix representation.

(a) $\Rightarrow$ (c): $\langle x, y\rangle = \langle T^*T(x), y\rangle$

$= \langle T(x), T(y)\rangle$