Today's Plan
1. Review
2. Spectral Theorem
3. Singular Value Decomposition (SVD)
4. Jordan Form

$\boxed{\text{Review}}$ Notation and facts:

- V is an inner product space over $F = \mathbb{C}$ (or $\mathbb{R}$).
- $T : V \to V$ is a linear operator
- $A \in F^{n \times n}$ is a matrix
- T^* is the adjoint operator satisfying

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \forall x, y \in V$$

$$\langle x, T(y) \rangle = \langle T^*(x), y \rangle$$

- If β is an orthonormal basis, then

$$[T^*]_\beta = [T]_\beta^*$$

- $(L_A)^* = L_{A^*}$

$(cT)^* = \bar{c}T^*$, $c \in F$

$(TU)^* = U^*T^*$

$(T^*)^* = T$

- If $B \in F^{m \times n}$, $x \in F^m$, $y \in F^n$

$$\underbrace{\langle x, By \rangle}_{\text{standard inner prod in } F^m} = \underbrace{\langle B^*x, y \rangle}_{\text{standard inner prod in } F^n}$$

$B^* \in F^{n \times m}$

- A normal operator : $TT^* = T^*T$

  A normal matrix : $AA^* = A^*A$

- A self-adjoint operator, A Hermitian operator : $T = T^*$, $F = \mathbb{R} / \mathbb{C}$

examples of normal:

A self-adjoint matrix : $A^* = A$, $A \in \mathbb{C}^{n\times n}$

A real symmetric matrix : $A^T = A$, $A \in \mathbb{R}^{n\times n}$

A Hermitian matrix : $A \in \mathbb{C}^{n\times n}$, $A = A^*$

A real skew-symmetric matrix : $A \in \mathbb{R}^{n\times n}$, $A = -A^*$

A skew-Hermitian matrix : $A \in \mathbb{C}^{n\times n}$, $A = -A^*$.

- An unitary operator: $\|T(x)\| = \|x\|$, $\forall x \in V$, $F = \mathbb{C}$

  An orthogonal operator: $\|T(x)\| = \|x\|$, $\forall x \in V$, $F = \mathbb{R}$.

  An unitary matrix : $AA^* = I = A^*A$, $A \in \mathbb{C}^{n\times n}$

  i.e., A has orthonormal $\begin{cases} \text{rows} \\ \text{cols} \end{cases}$

  $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle = \langle x, x \rangle = \|x\|^2$

  An orthogonal matrix: $AA^* = I = A^*A$, $A \in \mathbb{R}^{n\times n}$.

- For $\dim(V) = n$, T is diagonalizable if T has n linearly independent eigenvectors $v_i \in V$

  Then $\beta = \{v_1, \dots, v_n\}$ is a basis

  $$T[v_i] = \lambda v_i = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_{i-1} + \lambda_i v_i + 0 \cdot v_{i+1} + \cdots + 0 \cdot v_n$$

  $$\Rightarrow [T]_\beta = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- $A \in F^{n\times n}$ is diagonalizable if A has n linearly independent eigenvectors $v_i \in F^n$

  Then $Q = [v_1\ v_2 \cdots v_n]$ is invertible.

$$AQ = [Av_1\ Av_2 \cdots Av_n]$$
$$= [\lambda_1 v_1\ \lambda_2 v_2 \cdots \lambda_n v_n]$$
$$= [v_1 \cdots v_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = Q\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$
$$\Leftrightarrow A = Q\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}Q^{-1}$$

- $\left.\begin{matrix}\dim(V)=n,\ T \\ A \in F^{n\times n}\end{matrix}\right\}$ is diagonalizable if and only if

  ① n eigenvalues in F including repeated ones

  Ex: 1) $F = \mathbb{C}$, $\det(A-\lambda I)=0$ always has n complex roots.

  2) $F = \mathbb{R}$, assume $\begin{cases} A^T = A \\ T \text{ is self-adjoint}\end{cases}$, n real eigenvalues.

  Lemma before Theorem 6.17

  ② geo multiplicity = alg multiplicity.

Important Theorems

Theorem 6.15 If T is normal, then

① $\|T(x)\| = \|T^*(x)\|$, $\forall x \in V$.

② $T - cI$ is normal, $c \in F$.

③ $T(v) = \lambda v \Rightarrow T^*(v) = \bar{\lambda} v$

④ $\left.\begin{array}{l} T[v_1] = \lambda_1 v_1 \\ T[v_2] = \lambda_2 v_2 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Rightarrow \langle v_1, v_2 \rangle = 0$

This means eigenspaces are orthogonal to one another.

Theorem 6.16 $F = \mathbb{C}$, $\dim(V) = n$

T is normal $\Leftrightarrow$ T has n orthonormal eigenvectors v_i

In other words

$$TT^* = T^*T \iff \begin{cases} \beta = \{v_1, \dots, v_n\} \text{ is orthonormal} \\ [T]_\beta = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad [T^*]_\beta = \begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix} \end{cases}$$

Theorem 6.19 (Matrix Version of Theorem 6.16)

$A \in \mathbb{C}^{n \times n}$

A is normal $\Leftrightarrow$ A is unitarily equivalent to a diagonal matrix

In other words

$AA^* = A^*A \Leftrightarrow A$ has n orthonormal eigenvectors v_i

$(Q = [v_1 \cdots v_n]$ is invertible and $Q^{-1} = Q^*)$

$$\Leftrightarrow A = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1} = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^*$$

Theorem 6.17 (real version of Theorem 6.16)

$F = \mathbb{R}$, $\dim(V) = n$

T is self-adjoint $\Leftrightarrow$ T has n orthormal eigenvector $v_i \in V$.

$$T = T^* \Leftrightarrow \begin{cases} \beta = \{v_1, \dots, v_n\} \text{ is orthonormal} \\ [T]_\beta = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \end{cases}$$

Here we can have n eigenvalues in $\mathbb{R}$ only when T is self-adjoint.

Theorem 6.20 (Matrix Version of Theorem 6.17) or real version of Theorem 6.19

$A \in \mathbb{R}^{n\times n}$

A is real symmetric $\Leftrightarrow$ A is orthogonally equivalent to a diagonal matrix

$$A = A^T \Leftrightarrow \begin{cases} A \text{ has } n \text{ real eigenvalues } \lambda_i \\ A \text{ has } n \text{ orthonormal eigenvectors } v_i \end{cases}$$

$\left(Q = [v_1 \cdots v_n] \in \mathbb{R}^{n\times n} \text{ is invertible and } Q^{-1} = Q^T \right)$

$$\Leftrightarrow A = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1} = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^T$$

all numbers are real!

Proof: HW.

Spectral Theorem (Matrix Version)

① If $A \in \mathbb{C}^{n\times n}$ is normal, let $\lambda_1, \dots, \lambda_k$ be all its distinct eigenvalues with eigenspaces $E_{\lambda_1}, \dots, E_{\lambda_k}$, $1 \le k \le n$

then eigenspaces are orthogonal to one another.

and $\sum_{i=1}^{k} \dim(E\lambda_i) = n$ (because it's diagonalizable)

$1 \le k \le n$

② If $A \in \mathbb{R}^{n\times n}$ is symmetric, let $\lambda_1, \dots, \lambda_k$ be all its distinct eigenvalues with eigenspaces $E\lambda_1, \dots, E\lambda_k$, then eigenspaces are orthogonal to one another.

and $\sum_{i=1}^{k} \dim(E\lambda_i) = n$ (because it's diagonalizable)

Singular Value Decomposition (SVD)

Theorem ① Any matrix $A \in \mathbb{C}^{n\times n}$ has a SVD:

$$A = U \underbrace{\Sigma}_{\text{diagonal}} V^* = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^*$$

where 1) $\sigma_i \geq 0$ are real, called singular values

and σ_i^2 are eigenvalues of AA^* (or A^*A)

it can be proven they have the same eigenvalues

$B = AA^*$, then $B^* = (AA^*)^* = AA^* = B$

So B is Hermitian thus eigenvalues $\in \mathbb{R}$

Lemma before Theorem 6.17

We can further show eigenvalues of B are nonnegative from the fact that $x^* B x = x^* A A^* x = \langle A^* x, A^* x \rangle \geq 0$.

Let $\lambda(AA^*)$ be eigenvalues of AA^*, then $\sigma(A) = \sqrt{\lambda(AA^*)} \geq 0$ for any $A \in \mathbb{C}^{n\times n}$.

2) U and V are unitary matrices.

and cols of U are orthonormal eigenvectors of AA^*

cols of V are orthonormal eigenvectors of A^*A.

② Any matrix $A \in \mathbb{R}^{n\times n}$ has a SVD:

$$A = U\Sigma V^T = U\begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} V^T$$

where 1) $\sigma_i \geq 0$ are singular values

and $\sigma_i(A) = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)} \geq 0$

always real non-negative

2) U and V are orthogonal matrices. $\in \mathbb{R}^{n\times n}$

and cols of U are orthonormal eigenvectors of AA^T

cols of V are orthonormal eigenvectors of A^TA

Example: Find SVD of $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

① $A^TA = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

eigenvalues of A^TA are $9, 4, 0$

$\Rightarrow$ singular values of A are $3, 2, 0$

② Find orthonormal eigenvectors of A^TA, form V.

③ Find orthonormal eigenvectors of AA^T, form U.

④ Verify $A = U\begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} V^T$.

Jordan Canonical Form of $A \in \mathbb{C}^{n\times n}$

What if we don't have n independent eigenvectors?

Define generalized eigenvector:

eigenvector : $Av = \lambda v \Leftrightarrow (A-\lambda I)v = 0$

generalized eigenvector : $\begin{cases} \boxed{Au_1 = \lambda u_1 + v} \Leftrightarrow (A-\lambda I)u_1 = v \\ Au_2 = \lambda u_2 + u_1 \Leftrightarrow (A-\lambda I)u_2 = u_1 \end{cases}$

if v is an eigenvector of λ, then u_1, u_2 are generalized eigenvectors.

Example: $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 6 & 3 & 2 & 0 & 0 \\ 10 & 6 & 3 & 2 & 0 \\ 15 & 10 & 6 & 3 & 2 \end{bmatrix}$

We don't have n independent eigenvectors

But we always have n independent eigenvectors and generalized eigenvectors $v_1, \dots, v_n$

such that

$$Q = [v_1 \ v_2 \cdots v_n]$$

$$A = Q \underbrace{J}_{\text{Jordan Form}} Q^{-1}$$

$A = QJQ^{-1} \Leftrightarrow AQ = QJ$

First col of $AQ = Av_1 = \lambda_1 v_1 = v_1$

Second col $\cdots = Au_1 = \lambda_1 u_1 + v_1$
$= u_1 + v_1$

First col of $QJ = v_1$

Second col of $QJ = v_1 + u_1$

For this example $\lambda_1 = 1$, $\lambda_2 = 2$

$$A = Q \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} Q^{-1}$$

① For λ_1, alg mul $=2$
geo mul $= 1$
$Av_1 = \lambda_1 v_1$
$Au_1 = \lambda u_1 + v_1$

② For λ_2, alg mul $=3$
geo mul $= 1$
$Av_2 = \lambda_2 v_2$
$Au_2 = \lambda_2 u_2 + v_2$
$Au_3 = \lambda_2 u_3 + u_2$

$Q = [v_1\ u_1\ v_2\ u_2\ u_3]$

In general

$$J = \begin{bmatrix} \boxed{\Lambda_1} & & & \\ & \boxed{\Lambda_2} & & \\ & & \ddots & \\ & & & \boxed{\Lambda_k} \end{bmatrix}$$

Assume k distinct eigenvalues, the J is a block diagonal with k blocks Λ_i

$1 \le$ geo mul $\le$ alg mul

Each Λ_i looks like (examples)

$\begin{cases} \text{alg mul} = 3 \\ \text{geo mul} = 3 \end{cases} \Rightarrow$ 1) $\Lambda_i = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & \lambda_i \end{pmatrix}$

$\begin{cases} \text{alg mul} = 4 \\ \text{geo mul} = 2 \end{cases} \Rightarrow$ 2) $\Lambda_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}$

or

$\Lambda_i = \begin{pmatrix} \lambda_i & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}$

or

$\Lambda_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}$

$\Big\{$ alg mul $-$ geo mul $=$ numbers of one in Λ_i

alg mul = 3
geo mul = 1 $\Big\} \Rightarrow$ 3) $\Lambda_i = \begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix}$

alg mul = 4
geo mul = 1 $\Big\} \Rightarrow$ 4) $\Lambda_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}$

Matrix Exponential : $A \in \mathbb{C}^{n \times n}$

want to define $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$

because

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

If A is diagonalizable, $A = QDQ^{-1}$

$$A^2 = A \cdot A = Q D \underline{Q^{-1} Q} D Q^{-1} = Q \underline{DD} Q^{-1}$$

$$= \underline{Q} \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix} \underline{Q^{-1}}$$

$$A^3 = \underline{Q} \begin{pmatrix} \lambda_1^3 & & \\ & \ddots & \\ & & \lambda_n^3 \end{pmatrix} \underline{Q^{-1}}$$

$$\Rightarrow e^A = I + A + \frac{A^2}{2!} + \cdots$$

$$= Q \left[\underline{I} + \underline{D} + \underline{\frac{D^2}{2!}} + \underline{\frac{D^3}{3!}} + \cdots \right] Q^{-1}$$

$$= Q \begin{bmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\frac{\lambda_1^3}{3!}+\cdots & & \\ & \ddots & \\ & & \end{bmatrix} Q^{-1}$$

$$= Q \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} Q^{-1}$$