Remark on V.S.:

V is a set with $\oplus$ and $\odot$

F is a set with addition and multiplication.

could be unrelated

Ex: $V = \{$all continuous functions$\}$

$F = \mathbb{R}$.

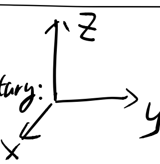
could be unrelated

Ex: HW#1 P5.

Linear Combination $a_i \in F$, $v_i \in V$

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + \cdots$$

Elementary:



Algebra: abstract V.S.

Linear Independence

$$a_1 v_1 + \cdots + a_n v_n = \vec{0} \Rightarrow a_i = 0, \forall i.$$

Remark: If $\vec{0} \in S$, then S is dependent.

$$S = \{\vec{0}, v_1, \cdots, v_n\} \Rightarrow 0 \cdot v_1 + \cdots + 0 \cdot v_n + \underline{1} \cdot \vec{0} = \vec{0}$$

Theorem 1.2

Theorem 1.6 $S_1 \subseteq S_2 \subseteq V$

① S_1 is dependent $\Rightarrow$ S_2 is dependent

② S_2 is independent $\Rightarrow$ S_1 is independent.

A if and only if B
iff

① "if" $B \Rightarrow A$

② "only if" $A \Rightarrow B$

If $A \Rightarrow B$, then not $B \Rightarrow$ not A

contrapositive

Section 1.6 Basis & Dimension

Elementary: $\mathbb{R}^3$

Abstract:

Def A basis β for V is a linearly independent subset of V and $\mathrm{Span}(\beta) = V$.

Vectors in β are called basis vectors.

Ex 1: Convention $\mathrm{Span}\{\phi\} = \{\vec{0}\}$

$\{\vec{0}\}$ is a V.S. over F.

ϕ is a basis for $\{\vec{0}\}$.

Ex 2: In $F^{n\times 1}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

standard basis.

Ex 3: $P_n(F) = \{$single variable polynomial of degree at most n with coefs in $F\}$ is a V.S. over F.

① $\forall f(x) \in P_n(F)$, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

② If $b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 = \vec{0}$

then $b_i = 0$, $\forall i$.

$S = \{1, x, x^2, \cdots, x^n\}$ is a basis.

Ex 4: $P(F) = \{$single variable polynomial with coefs in $F\}$

$\{1, x, x^2, \cdots, x^n, \cdots\}$ is a basis.

Theorem 1.8 $\beta = \{u_1, \cdots, u_n\}$ is a basis

iff each $v \in V$ has a unique expression

as $v = a_1 u_1 + \cdots + a_n u_n$, $a_i \in F$.

Proof: "if" We already have $\mathrm{Span}(\beta) = V$.

Only need to show β is independent.

$a_1 u_1 + \cdots + a_n \vec{u}_n = \vec{0} \Rightarrow a_i = 0, \forall i$ because of $\begin{cases} \vec{0} = 0 \cdot u_1 + \cdots + 0 \cdot u_n \\ \text{uniqueness of expression.} \end{cases}$

$\Rightarrow \{u_1, \ldots, u_n\}$ is independent.

"only if" β is a basis $\Rightarrow \text{Span}(\beta) = V \Rightarrow$

$\forall v \in V$, $\underline{v = a_1 u_1 + \cdots + a_n u_n}$ ①

Only need to show uniqueness.

Assume the same $\underline{v = b_1 u_1 + \cdots + b_n u_n}$ ②

(VS4) $\underline{① - ②} \Rightarrow \vec{0} = (a_1 - b_1) u_1 + \cdots + (a_n - b_n) u_n$

$\Downarrow$ linear inpendence of β

$a_i - b_i = 0, \forall i$

$\Downarrow$

uniqueness.

② $\Rightarrow -v = -(b_1 u_1 + \cdots + b_n u_n)$

$\Rightarrow (-1) \cdot v = (-1)(b_1 u_1 + \cdots + b_n u_n)$

$= (-b_1) u_1 + \cdots + (-b_n) u_n$ ③

① + ③ $\Rightarrow v + (-1) v = a_1 u_1 + \cdots + a_n u_n + (-b_1) u_1 + \cdots + (-b_n) u_n$

$\Rightarrow \vec{0} = (a_1 - b_1) u_1 + \cdots + (a_n - b_n) u_n$

Theorem 1.9 If $V = \text{Span}(S)$ and S is a finite set then some subset of S is a basis of V.

Proof: ① If $S = \phi$ or $S = \{\vec{0}\}$, then $V = \{\vec{0}\}$

Then $\phi \subseteq S$ is a basis for $V = \{\vec{0}\}$ because $\text{Span}(\phi) = \{\vec{0}\}$.

② If S has at least one nonzero vector u_1, Let $\beta = \{u_1\}$. Look for another nonzero vector $u_2 \in S$

add u_2 to β if u_1, u_2 are independent.

$$\beta = \{u_1, u_2\}$$

Continue this process until done.

$$\beta = \{u_1, \dots, u_k\}$$

Next, want to β is a basis.

First, need to show $\underline{S \subseteq \text{Span}(\beta)}$.

1) $\forall v \in S$, if $v \in \beta$, then $v \in \text{Span}(\beta)$.

2) $\forall v \in S$, if $v \notin \beta$, want to show $\underline{v \in \text{Span}(\beta)}$

By construction of β, $v \notin \beta \Rightarrow \underline{\beta \cup \{v\}}$ is dependent.

$\{u_1, \dots, u_k, v\}$ dependent

$$\Rightarrow \exists\ (a_1, \dots, a_k, b) \neq (0, \dots, 0, 0)$$

s.t. $a_1 u_1 + \dots + a_k u_k + b v = \vec{0}$.

Assume $b = 0$, then $a_1 u_1 + \dots + a_k u_k = \vec{0}$

$$(a_1, \dots, a_k, 0) \neq (0, \dots, 0, 0)$$

$\Downarrow$

$u_1, \dots, u_k$ are dependent.

$\Downarrow$

Contradiction to the definition of β.

So $b \neq 0$.

Then $\underline{-bv = a_1 u_1 + \dots + a_k u_k}$

$$v = -\frac{a_1}{b} u_1 - \dots - \frac{a_k}{b} u_k.$$

Second, want to show $S \subseteq \text{Span}(\beta) \xRightarrow{\text{Theorem 1.5}} V = \text{Span}(S) \subseteq \text{Span}(\beta)$

$$\left.\begin{array}{r} V \subseteq \text{Span}(\beta) \\ \text{closedness} \xRightarrow[\text{Span}(\beta)]{} \text{Span}(\beta) \subseteq V \end{array}\right\} \Rightarrow V = \text{Span}(\beta).$$

Theorem 1.5 Any subspace containing S also contains $\mathrm{Span}(S)$

Theorem 1.10 (Replacement Theorem)

Let V be a vector space generated by G with n vectors. Let $L \subseteq V$ be linearly independent with m vectors.

Then, 1) $m \leq n$, and 2) $\exists H \subseteq G$ with $(n-m)$ vectors s.t. $L \cup H$ generates V.

Proof: ① $m=0$. Then $L = \phi$ and $H = G$.

② Assume it's true for $m \geq 0$, thus if $L = \{v_1, \cdots, v_m\}$ is independent, then 1) $m \leq n$ 2) $\exists H = \{u_1, \cdots, u_{n-m}\} \subseteq G$

(*) s.t. $L \cup H$ Generates V.

Want to show it's true for $m+1$, i.e.,

if $\bar{L} = \{\bar{v}_1, \cdots, \bar{v}_m, \bar{v}_{m+1}\}$ is independent, then 1) $m+1 \leq n$ and 2) $\exists \bar{H} = \{\bar{u}_1, \cdots, \bar{u}_{n-m-1}\} \subseteq G$ s.t. $\bar{L} \cup \bar{H}$ generates V.

Proof of 1): want to show $m+1 \leq n$.

assume $m+1 > n$.

$(*) \Rightarrow m \leq n \Rightarrow m \leq n < m+1 \Rightarrow n = m.$

$\Rightarrow n-m=0 \Rightarrow H = \phi$ in $(*)$

$(*) \Rightarrow \{\bar{v}_1, \cdots, \bar{v}_m\} \cup \phi$ generates V.

$\Rightarrow \bar{v}_{m+1} \in V = \text{Span}(\{\bar{v}_1, \cdots, \bar{v}_m\} \cup \phi)$

$\Rightarrow \bar{v}_{m+1} \in \text{Span}\{\bar{v}_1, \cdots, \bar{v}_m\}$

$\Rightarrow \bar{v}_1, \cdots, \bar{v}_m, \bar{v}_{m+1}$ are dependent

$\Rightarrow$ Contradiction.

Proof of 2): Since $m+1 \leq n$, $n-m \geq 1$.

Apply $(*)$ to $\underbrace{\{\bar{v}_1, \cdots, \bar{v}_m\}}_{L \text{ in } (*)}$, then

$\exists\ H = \{u_1, \cdots, u_{n-m}\} \subseteq G$ s.t.

$V = \text{Span}\{\bar{v}_1, \cdots, \bar{v}_m, u_1, \cdots, u_{n-m}\}$

$\bar{v}_{m+1} \in V \Rightarrow \bar{v}_{m+1} = a_1\bar{v}_1 + \cdots + a_m\bar{v}_m + b_1u_1 + \cdots + b_{n-m}u_{n-m}$

$\Rightarrow a_1\bar{v}_1 + \cdots + a_m\bar{v}_m - \bar{v}_{m+1} + b_1u_1 + \cdots + b_{n-m}u_{n-m} = \vec{0}$

Claim at least one $b_i \neq 0$.

Proof of claim: if not, then $b_i = 0, \forall i$.

$\Rightarrow a_1\bar{v}_1 + \cdots + a_m\bar{v}_m - \bar{v}_{m+1} = \vec{0}$

$\Rightarrow L = \{\bar{v}_1, \cdots, \bar{v}_m, \bar{v}_{m+1}\}$ dependent

$\Rightarrow$ Contradiction.

So at least one of $b_i \neq 0$.

Without loss of generality (WLOG), assume $b_1 \neq 0$.

$$-b_1 u_1 = a_1\bar{v}_1 + \cdots + a_m\bar{v}_m - \bar{v}_{m+1} + b_2 u_2 + \cdots + b_{n-m}u_{n-m}$$

$$u_1 = -\frac{a_1}{b_1}\bar{v}_1 - \cdots - \frac{a_m}{b_1}\bar{v}_m + \frac{1}{b_1}\bar{v}_{m+1} - \frac{b_2}{b_1}u_2 - \cdots - \frac{b_{n-m}}{b_1}u_{n-m}$$

$$\Rightarrow u_1 \in \mathrm{Span}\{\underbrace{\bar{v}_1, \cdots, \bar{v}_m, \bar{v}_{m+1}}_{\bar{L}}, \underbrace{u_2, \cdots, u_{n-m}}_{\bar{H}}\}$$

$$= \mathrm{Span}(\bar{L} \cup \bar{H}) \Rightarrow L \cup H \subseteq \mathrm{Span}(\bar{L} \cup \bar{H})$$

Theorem 1.5 $\underbrace{\text{Any subspace}}_{\mathrm{Span}(\bar{L}\cup\bar{H})}$ containing a set $\underbrace{S}_{L \cup H}$ also contains $\underbrace{\mathrm{Span}(S)}_{\mathrm{Span}(L\cup H) = V}$.

$$\Rightarrow \left.\begin{matrix} V \subseteq \mathrm{Span}(\bar{L} \cup \bar{H}) \\ \mathrm{Span}(\bar{L} \cup \bar{H}) \subseteq V \end{matrix}\right\} \Rightarrow V = \mathrm{Span}(\bar{L} \cup \bar{H}).$$

Corollary 1: V has a finite basis, then all bases are finite with the same number of vectors.

Proof: Assume β is a basis with n vectors

Let S be any basis with m vectors.

Theorem 1.10 $\left.\begin{matrix} L & m \text{ vectors} \\ G & n \text{ vectors} \end{matrix}\right\} \Rightarrow m \leq n$.

$$\left.\begin{matrix} G = \beta & n \\ L = S & m \end{matrix}\right\} \Rightarrow m \leq n.$$

$$\left.\begin{matrix} G = S & m \\ L = \beta & n \end{matrix}\right\} \Rightarrow n \le m$$

Def: V is finite-dimensional if V has a finite basis with n vectors. $\dim(V) = n$.

If V does not have a finite basis, then V is infinite-dimensional.

Ex: ① $\mathbb{R}^3$, $\dim = 3$.

② $V = \{\text{all continuous functions}\}$

$P(F) = \{\text{all polynomials}\}$

$V = \{(a_1, \cdots, a_n, \cdots), a_i \in F\}$

③ $V = \{\vec{0}\}$ $\dim$ is 0.

$= \mathrm{Span}(\phi)$

$\mathrm{Span}(\{\vec{0}\}) = \{\vec{0}\}$

Is $\{\vec{0}\}$ a basis for $V = \{\vec{0}\}$?

No: $a\vec{0} = \vec{0}$

④ F^n $\dim$ is n

$F^{m \times n}$ $\dim$ is mn

$P_n(F)$ $\dim$ is $n+1$

$\{1, x, x^2, \cdots, x^n\}$

⑤ $V = \mathbb{C}$ over $F = \mathbb{C}$: $\dim$ is 1

$\{1\}$

⑥ $V = \mathbb{C}$ over $F = \mathbb{R}$: $\dim$ is 2

$\forall x \in V$, $x = a + ib = \text{Span}\{1, i\}$

$a, b \in F = \mathbb{R}$.

Corollary 2 : V has dim n.

(a) If $\text{Span}(S) = V$, S has n vectors, then S is a basis.

(b) Any independent set with n vectors is a basis.

(c) Every independent set L can be extended to a basis.

Ex: $$a\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} + d\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\-1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$\Rightarrow$ independant $\Rightarrow$ a basis for $\mathbb{R}^4$.

Theorem 1.11

W is subspace of V and $\dim(V)$ is finite.

$\Rightarrow \dim(W) \le \dim(V)$

If $\dim(W) = \dim(V)$, then $W = V$.

Chapter 2 Linear Transformation and Matrices

Def V and W are two V.S. over the same F.

A mapping $T: V \to W$ is a linear transformation

if $\forall x, y \in V$, $\forall c \in F$

① $T(x+y) = T(x) + T(y)$

$\oplus$ in V $\oplus$ in W

② $T(c \cdot x) = c \cdot T(x)$

$\odot$ in V $\odot$ in W

Properties:

① Def $\Rightarrow T(\vec{0}) = \vec{0}$

$\vec{0} \in V$ $\vec{0} \in W$.

② Def $\Leftrightarrow \begin{cases} \forall x, y \in V, \ \forall a, b \in F, \\ T(ax+by) = aT(x) + bT(y) \end{cases}$

③ Def $\Rightarrow T(x-y) = T(x) - T(y)$

④ Def $\Leftrightarrow T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$

$a_i \in F$, $x_i \in V$.

Ex: $T: \mathbb{R}^{1\times 2} \to \mathbb{R}^{1\times 2}$

$(a, b) \mapsto (2a+b, b)$

$T(a,b) = (2a+b, b)$

Verification of ①: $\forall x, y \in \mathbb{R}^{1\times 2}$,

want to show $T(x+y) = T(x) + T(y)$.

Let $x = (a_1, b_1)$, $y = (a_2, b_2)$, $LHS = T(x+y) = T(a_1+a_2, b_1+b_2)$

$$= \left(2(a_1+a_2)+(b_1+b_2),\ b_1+b_2\right)$$

$$RHS = T(x)+T(y)$$

$$= T(a_1, b_1) + T(a_2, b_2)$$

$$= (2a_1+b_1, b_1) + (2a_2+b_2, b_2)$$

Verify ②:

Ex: $f(x): \mathbb{R} \longrightarrow \mathbb{R}$ is NOT a linear transformation.

$$x \longmapsto x+1$$

$$f(0) = 1 \neq 0$$