Review

$T: V \to W$

- Dim Thm: $\dim(N(T)) + \dim(R(T)) = \dim(V)$
- Theorem: $\dim(V) = \dim(W) \Rightarrow$ equivalency of $\begin{cases} ① \text{ 1-on-1} \\ ② \text{ onto} \\ ③ \text{ Nullity}(T) = 0 \end{cases}$

$\Updownarrow$

$\dim(R(T)) = \dim(V)$

- $[v]_\beta$
- $T: \underset{\beta}{V} \to \underset{\gamma}{W}$ $\quad [T]_\beta^\gamma$

$v \mapsto T(v)$ $\quad [T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$.

- Theorem 2.19 $\quad V$ is isomorphic to $W \Leftrightarrow \dim(V) = \dim(W)$

There is $T: V \to W$
1-on-1
onto
invertible
T^{-1} is linear

- Theorem 2.20 $\quad \underset{\beta}{V}, \underset{\gamma}{W}$ are V.S. over F.

$\dim(V) = n \quad \dim(W) = m$

$$\Phi_\beta^\gamma : \mathcal{L}(V, W) \to F^{m \times n} \text{ is an isomorphism.}$$

$$T \longmapsto [T]_\beta^\gamma$$

- Corollary: $\dim(\mathcal{L}(V,W)) = \dim(F^{m \times n}) = mn$.

Ex: ① $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2) \sim F^{2 \times 3}$

is isomorphic to

② $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \sim F^{3 \times 2}$

④ $\mathcal{L}(V, W) \sim F^{n \times m}$

Thm 2.20 $\Rightarrow \mathcal{L}(V,W) \sim F^{m \times n}$

$\Rightarrow \dim(\mathcal{L}(V,W)) = \dim(F^{m \times n})$

$$= mn = \dim(F^{n\times m})$$

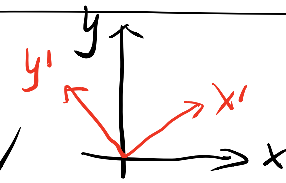
$$\Rightarrow \mathcal{L}(V,W) \sim F^{n\times m}$$

⑤ $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2) \sim \mathcal{L}(\mathbb{R}^2,\mathbb{R}^3)$

⑥ $\mathcal{L}(\underbrace{\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)}_{4}, \underbrace{\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)}_{4}) \sim \mathbb{R}^{4\times 4}$

Elements in $\underline{\mathcal{L}(\underline{\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)}, \mathbb{R})}$ are called functionals.

Change of Coordinates



Given $\left.\begin{aligned}\beta &= \{v_1, \dots, v_n\} \\ \gamma &= \{w_1, \dots, w_n\}\end{aligned}\right\}$ bases of V

Q: Given $[v]_\beta$, what is $[v]_\gamma$?

$$I_V : \underset{\beta}{V} \longrightarrow \underset{\gamma}{V}$$
$$v \longmapsto v$$

$$\underline{[v]_\gamma} = [I_V(v)]_\gamma = [I_V]_\beta^\gamma [v]_\beta$$

$$\left(\begin{aligned} &T : \overset{n}{\underset{\beta}{V}} \longrightarrow \overset{m}{\underset{\gamma}{W}} \qquad T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \\ &\quad v \longmapsto T(v) \\ &[T]_\beta^\gamma = (a_{ij})_{m\times n} \end{aligned} \right)$$

$$[I_V]_\beta^\gamma = \Big(\quad \underset{j\text{-th col}}{\boxed{}} \xrightarrow{} [I_V(v_j)]_\gamma \quad \Big)_{n\times n} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex: $P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2\}$

$\gamma = \{1, (x-1), (x-1)^2\}$

$$I_V(1) = 1 = \underline{1}\cdot 1 + \underline{0}\cdot(x-1) + \underline{0}\cdot(x-1)^2$$

$$I_V(x) = x = \underline{1}\cdot 1 + \underline{1}\cdot(x-1) + \underline{0}\cdot(x-1)^2$$

$$I_V(x^2) = x^2 = a\cdot 1 + b(x-1) + c(x-1)^2$$

Solve for $a, b, c \Rightarrow \begin{cases} a=1 \\ b=2 \\ c=1 \end{cases}$

$$= 1\cdot 1 + 2\cdot(x-1) + 1\cdot(x-1)^2$$

Given $f(x) = ax^2 + bx + c$

want $f(x) = A(x-1)^2 + B(x-1) + C$

$[f]_\beta = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$, $[f]_\gamma = \begin{bmatrix} C \\ B \\ A \end{bmatrix}$

$$[f]_\gamma = [I_V]_\beta^\gamma [f]_\beta$$

$$\begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{pmatrix} a+b+c \\ b+2a \\ a \end{pmatrix}.$$

Chapter 3 Linear System (Geometric Meaning of Sols)

Example: $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : 2a = b+1,\ b = 2c,\ 3c+6d = a \right\}$

$$\left.\begin{array}{l} 2a = b+1 \\ b = 2c \\ 3c+6d = a \end{array}\right\} \Rightarrow \left|\begin{array}{l} 2a-b=1 \\ b-2c=0 \\ -a+3c+6a=0 \end{array}\right. \Rightarrow \begin{cases} 2x_1 - x_2 + 0\cdot x_3 + 0\cdot x_4 = 1 \\ \quad\ \ x_2 - 2x_3 \qquad\quad = 0 \\ -x_1 \qquad + 3x_3 + 6x_4 = 0. \end{cases}$$

Step 0.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & 0 & 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ -1 & 0 & 3 & 6 & 0 \end{array}\right)$$ Augmented Matrix

Step I. Gaussian Elimination (Row Operations to get row echelon form)

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ -1 & 0 & 3 & 6 & 0 \end{array}\right)$$

$r1 \Longleftrightarrow r3$ (Type 1, switch 2 rows)

$$\Rightarrow \left(\begin{array}{cccc|c} -1 & 0 & 3 & 6 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$(-1)\cdot r1 \longrightarrow r1$ (Type 2, scalar mul to a row)

$$\Rightarrow \left(\begin{array}{cccc|c} \underline{1} & 0 & -3 & -6 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$(-2)\cdot r1 + r3$ (Type 3, add a scalar multiple of a row to another row)

$$\Rightarrow \left(\begin{array}{cccc|c} \underline{1} & 0 & -3 & -6 & 0 \\ 0 & \underline{1} & -2 & 0 & 0 \\ 0 & -1 & 6 & 12 & 1 \end{array}\right)$$

$r2 + r3 \rightarrow r3$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -3 & -6 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & \underline{4} & 12 & 1 \end{array}\right)$$

$\frac{1}{4}\cdot r3 \rightarrow r3$

$$\Rightarrow \left(\begin{array}{cccc|c} \underline{1} & 0 & -3 & -6 & 0 \\ 0 & \underline{1} & -2 & 0 & 0 \\ 0 & 0 & \underline{1} & 3 & 1/4 \end{array}\right)$$

Leading ones

Step II For cols w/o leading ones, the corresponding variables become parameter.

$$x_4 = t$$

Solve for the others.

$$\Rightarrow x_3 = \frac{1}{4} - 3x_4 = \frac{1}{4} - 3t$$

$$x_2 = 2x_3 = \frac{1}{2} - 6t$$

$$x_1 = 3x_3 + 6x_4 = \frac{3}{4} - 9t + 6t = \frac{3}{4} - 3t$$

Step III.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3/4 - 3t \\ 1/2 - 6t \\ 1/4 - 3t \\ t \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2 \\ 1/4 \\ 0 \end{pmatrix} + \begin{pmatrix} -3t \\ -6t \\ -3t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 \\ 1/2 \\ 1/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -6 \\ -3 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}.$$

Def Type 1, 2, 3 elementary matrices are generated by Type 1, 2, 3 row/col operations on I.

Ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r1 \leftrightarrow r2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad (-2)\cdot r3 + r1 \to r1$

Theorem 3.1 $A \in F^{m\times n}$. Let B be generated by a row (col) operation on A. Then there is a $m\times m$ ($n\times n$) elementary matrix E s.t.

$$B = EA. \quad (B = AE)$$

If $B = EA$, then B can be generated by a row operation on A.

Theorem 3.2 Elementary matrices are invertible. The inverse is elementary of the same type.

Def $A \in F^{m\times n}$, rank of A, denoted by rank(A) is the rank of $L_A: F^n \longrightarrow F^m$

$$x \longmapsto Ax$$

$\text{rank}(L_A)$
$\,\,\,\,||$
$\dim(R(L_A))$

$$\text{Rank}(A) \triangleq \text{Rank}(L_A) = \dim(R(L_A))$$
$$= n - \dim(N(L_A))$$

Theorem 3.3 $T: \underset{n}{\overset{\beta}{V}} \longrightarrow \underset{m}{\overset{\gamma}{W}}$

$$\operatorname{rank}(T) = \operatorname{rank}([T]_\beta^\gamma).$$

Proof: Let $A = [T]_\beta^\gamma$.

$$L_A : F^n \longrightarrow F^m$$
$$x \longmapsto Ax$$

$\phi_\beta : V \to F^n$
$v \longmapsto [v]_\beta$
is an isomorphism

Independence of β
$\Rightarrow [v]_\beta = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ iff $v = \vec{0}$.

$N(\phi_\beta) = \{\vec{0}\}$
$\Rightarrow \phi_\beta$ is 1-one, onto
$\Rightarrow \phi_\beta$ is invertible
$\Rightarrow \phi_\beta$ is an isomorphism.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ F^n \arrow[r, "L_A"] & F^m \end{tikzcd}$$

$\phi_\gamma : W \to F^m$
$w \longmapsto [w]_\gamma$
is isomorphism

Diagram Commutes
$L_A \phi_\beta = \phi_\gamma T$

① $L_A \phi_\beta : V \longrightarrow F^m$

$$v \longmapsto L_A(\phi_\beta(v)) = L_A([v]_\beta) = \underline{[T]_\beta^\gamma [v]_\beta}$$

② $\phi_\gamma T : V \longrightarrow F^m$

$$v \longmapsto \phi_\gamma(T(v)) = [T(v)]_\gamma = \underline{[T]_\beta^\gamma [v]_\beta}$$

HW#3 pg $T: V \to W$ is isomorphism
V_0 is a subspace of V.
$\Rightarrow \begin{cases} T(V_0) \text{ is a subspace of } W. \\ \dim(V_0) = \dim(T(V_0)) \end{cases}$

Hint: $\tilde{T} : V_0 \to T(V_0)$
$v \longmapsto T(v)$

P9 means isomorphism can be removed when counting dimension of a subspace.

$$L_A \phi_\beta = \phi_\gamma T \Rightarrow N(L_A \phi_\beta) = N(\phi_\gamma T)$$

$$\Rightarrow \dim(N(L_A \phi_\beta)) = \dim(N(\phi_\gamma T))$$

$$P9 \Rightarrow \dim(N(T)) = \dim(\phi_\gamma(N(T)))$$

$V_0 = N(T)$

$\phi_\gamma : N(T) \to \phi_\gamma(N(T)) \Rightarrow \dim(N(T)) = \dim(\phi_\gamma(N(T)))$

$$\Rightarrow \dim(N(L_A)) = \dim(N(T))$$

$$(\dim \text{ Thm}) \Rightarrow \operatorname{rank}(L_A) = \operatorname{rank}(T)$$

$$\operatorname{rank}(L_A) = \operatorname{rank}(A)$$

Want to show $\dim(N(T)) = \dim(N(\phi_\gamma T))$

$N(\phi_\gamma T) \subseteq V$ $\quad \phi_\gamma T : V \to F^m$

$$N(\phi_\gamma T) = \phi_\gamma(N(T)) \ ?$$

$\phi_\gamma : V \to F^m$

$\phi_\gamma : N(T) \to \phi_\gamma(N(T))$

$\dim(N(T)) = \dim(\phi_\gamma(N(T)))$