Theorem 2.17 $T: V \to W$ is linear and invertible (1-on-1 and onto)
$\Rightarrow T^{-1}: W \to V$ is linear

Theorem 2.4 A linear $T: V \to W$ is 1-on-1 $\Leftrightarrow N(T) = \{\vec{0}\}$
Nullity $(T) = 0$

Theorem 2.5 If $\dim(V) = \dim(W)$ is finite, then
1-on-1 $\Leftrightarrow$ onto $\Leftrightarrow N(T) = \{\vec{0}\} \Leftrightarrow \dim(R(T)) = \dim(W)$
$\Leftrightarrow R(T) = W$ (onto)
(Dimension Thm $\dim(N(T)) + \dim(R(T)) = \dim(V)$)
$\dim(V) = \dim(W) \Rightarrow \dim(R(T)) = \dim(W)$
$\dim(N(T)) = 0$

Definition : If $T: V \to W$ is linear and invertible,
T is called an isomorphism.

Remark : Isomorphism is also defined for infinite-dimension vector spaces.

Theorem 2.19 If V and W are finite dimensional over same F,
then $V \sim W \Leftrightarrow \dim(V) = \dim(W)$.
is isomorphic to.

Corollary: If $\dim(V) = \dim(W)$ (is finite), $T: V \to W$ is linear.
then T is an isomorphism $\Leftrightarrow N(T) = \{\vec{0}\}$

Proof: By Theorem 2.5, $N(T) = \{\vec{0}\} \Leftrightarrow \left\{\begin{array}{l}\text{1-on-1} \\ \text{onto}\end{array}\right. \Leftrightarrow$ invertible.

Example: V has an ordered basis β with $\dim(V) = n$
$$\phi_\beta : V \longrightarrow F^n \text{ is linear.}$$
$$v \longmapsto [v]_\beta$$
Linear independence of $\beta \Rightarrow [v]_\beta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ if and only if $v = \vec{0}$.

So $N(\phi_\beta) = \{\vec{0}\}$. Since $\dim(V) = \dim(F^n)$, ϕ_β is an isomorphism.

9. (10 pts) Let V, W be finite dimensional vector spaces and $T : V \longrightarrow W$ be isomorphism. Let V_0 be a subspace of V. Prove that $T(V_0)$ is a subspace in W and $dim(V_0) = dim(T(V_0))$.

Solution: First, since V_0 is a subspace, $\vec{0}_V \in V_0$ thus $\vec{0}_W = T(\vec{0}_V) \in T(V_0)$. Second, for any $x, y \in T(V_0)$, any $a, b \in F$, there exist $u, v \in V_0$ s.t. $T(u) = x, T(v) = y$, and $ax+by = aT(u)+bT(v) = T(au+bv) \in V_0$. So $T(V_0)$ is a subspace.

Finally, we can give the restriction of T on V_0 a new name, call it $\bar{T}$. In other words, it is defined as:

$$\begin{aligned} \bar{T} : V_0 &\longrightarrow T(V_0) \\ v &\longmapsto T(v) \end{aligned}$$

Since T is an isomorphism, it is one-on-one thus $\bar{T}$ is also one-on-one. Obviously $\bar{T}$ is onto since $T(V_0)$ is its range. Since T is linear, $\bar{T}$ is linear. Thus $\bar{T}$ is invertible and linear, thus an isomorphism. By Theorem 2.19, $dim(V_0) = dim(T(V_0))$.

Example: W has an ordered basis γ with $\dim(W) = m$.

$\phi_\gamma : W \longrightarrow F^m$ is an isomorphism

$T : V \to W$ is some linear transformation

Then $R(T) \subseteq W$ is a subspace of W

Problem 9 $\Rightarrow$ $\dim(R(T)) = \dim(\phi_\gamma[R(T)])$

What is $\phi_\gamma[R(T)]$? $= \dim(R(\phi_\gamma T))$

$\phi_\gamma[R(T)] = \{\phi_\gamma(T(v)) : \forall v \in V\}$

Then what is $R[\phi_\gamma T]$? Range of $\phi_\gamma T$!

$R[\phi_\gamma T] = \{\underbrace{\phi_\gamma T(v)}_{= \phi_\gamma(T(v))} : \forall v \in V\}$

$\Rightarrow \dim(R(T)) = \dim(R(\phi_\gamma T))$

$\forall v \in V$, $\phi_\gamma T(v) = \phi_\gamma(T(v)) = [T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$ (Theorem)

$L_A \phi_\beta(v) = L_A(\phi_\beta(v)) = L_A[v]_\beta = A[v]_\beta = [T]_\beta^\gamma [v]_\beta$

Diagram Commutes

$\phi_\gamma T = L_A \phi_\beta$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd}$$

$A = [T]_\beta^\gamma$

$L_A : F^n \longrightarrow F^m$

$x \longmapsto Ax$

Example: $V = P_3(\mathbb{R})$, $W = P_2(\mathbb{R})$, T is differentiation $\frac{d}{dx}$

$T : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2, x^3\}$

$p(x) \longmapsto \frac{d}{dx} p(x) = p'(x)$ $\quad \gamma = \{1, x, x^2\}$

$p(x) = a + bx + cx^2 + dx^3$

$\frac{d}{dx} p(x) = b + 2cx + 3dx^2$

$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = A$

$L_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto Ax = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Want to show $\text{Rank}(T) = \text{Rank}(L_A)$

$\Updownarrow$ definition of rank of A

$\dim(R(T)) = \dim(R(L_A))$

(Problem 9) $\Updownarrow$

$\dim(R(\phi_\gamma T)) = \dim(R(L_A))$

(Diagram Commutes) $\Updownarrow$

$\dim(R(L_A \phi_\beta)) = \dim(R(L_A))$

$\Uparrow$

$R(L_A) = \{ L_A x : \forall x \in F^n \}$

$R(L_A \phi_\beta) = \{ L_A[\phi_\beta(v)] : \forall v \in V \}$

(ϕ_γ is 1-on-1 and onto) $= \{ L_A x : \forall x \in F^n \}$

$\Rightarrow R(L_A) = R(L_A \phi_\beta)$

Def Rank of a matrix $A \in F^{m\times n}$ is defined as the rank of L_A, i.e., the dimension of range of $L_A : F^n \to F^m$
$x \mapsto Ax$

Theorem 3.3 $T: \overset{\beta}{V} \to \overset{\gamma}{W}$
$\dim(V)=n \quad \dim(W)=m$

$\Rightarrow \underline{\operatorname{rank}(T)} = \underline{\operatorname{rank}([T]_\beta^\gamma)}$

Def dim of range of T
$\dim(R(T))$

Theorem 3.4 $\forall A \in F^{m\times n}$, if $P \in F^{m\times m}$, $Q \in F^{n\times n}$ are invertible, then

① $\operatorname{rank}(AQ) = \operatorname{rank}(A)$

② $\operatorname{rank}(PA) = \operatorname{rank}(A)$

③ $\operatorname{rank}(PAQ) \overset{①}{=} \operatorname{rank}(PA) \overset{②}{=} \operatorname{rank}(A)$

Proof: ① Q is invertible $\Rightarrow L_Q : F^n \to F^n$ is isomorphism
$x \mapsto Qx$

$\Rightarrow L_Q$ is onto $\Rightarrow L_Q(F^n) = F^n$

$L_{AQ} : F^n \to F^m$, $x \mapsto AQx$ | $L_A : F^n \to F^m$, $x \mapsto Ax$

$\Rightarrow L_{AQ} = L_A L_Q$

$$\begin{aligned}\operatorname{rank}(AQ) &= \dim(R(L_{AQ})) \\ &= \dim(R(L_A L_Q)) \\ &= \dim(L_A[L_Q(F^n)])\end{aligned}$$

$= \dim(L_A(F^n))$

$= \dim(R(L_A))$

$= \operatorname{rank}(A)$

$L_P : F^m \to F^m$

$x \mapsto Px$

② P is invertible $\Rightarrow$ L_P is isomorphism

(Problem 9) $\Rightarrow$ $\dim(V_0) = \dim(L_P(V_0))$ for any subspace V_0 in F^m.

In particular, let $V_0 = R(L_A)$

$$\Rightarrow \dim(R(L_A)) = \dim(L_P(R(L_A)))$$
$$= \dim(L_P[L_A(F^n)])$$
$$= \dim(L_P L_A(F^n))$$
$$= \dim(R(L_P L_A))$$
$$= \dim(R(L_{PA}))$$
$$= \operatorname{rank}(PA)$$

Corollary: Row and Col ops are rank-preserving.

Theorem 3.5: $A \in F^{m\times n}$, Rank(A) is equal to dimension of column space of A

a subspace in F^m spanned by all columns of A.

Proof: Rank$(A) = \dim(R(L_A))$

$= \dim\{L_A(x) : \forall x \in F^n\}$

$= \dim\{Ax : \forall x \in F^n\}$

$$\underbrace{\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = x_1\vec{c}_1 + \cdots + x_n\vec{c}_n$$

$= \dim\{x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n,\ \forall x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in F^n\}$

$= \dim(\text{col space of } A)$

Theorem 3.6 $A \in F^{m\times n}$ has rank r. Then $r \le m$, $r \le n$.

And A can be transformed to $\left(\begin{array}{c|c} \boxed{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}} & 0 \\ \hline 0 & 0 \end{array}\right)$ by row and col ops.

$I_{r\times r}$

echelon form

Proof: Theorem 3.4 $\Rightarrow$ Echelon form has the same rank.

$\dim(\text{col space of echelon form}) = r$

$\Rightarrow r =$ number of leading ones.

Theorem 3.7 ① $T: \underset{\alpha}{V} \to \underset{\beta}{W}$ $[T]_\alpha^\beta$ $\quad \text{rank}(UT) \le \text{rank}(U)$

$U: \underset{\beta}{W} \to \underset{\gamma}{Z}$ $[U]_\beta^\gamma$ $\quad \text{rank}(UT) \le \text{rank}(T)$

$UT: \underset{\alpha}{V} \to \underset{\gamma}{Z}$

$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

② $A \in F^{m\times n}$ $\quad \text{rank}(AB) \le \text{rank}(A)$

$B \in F^{n\times p}$ $\quad \text{rank}(AB) \le \text{rank}(B)$.

Proof: read book.

Computing A^{-1} by Gaussian Elimination

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$$

Step I: $(A \mid I)$

$$\left(\begin{array}{ccc|ccc} \underline{0} & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step II : Row ops to obtain echelon form for the left part.

$$r1 \leftrightarrow r2 \quad \left(\begin{array}{ccc|ccc} 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\tfrac{1}{2} r1 \rightarrow r1 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$(-3) r1 + r3 \rightarrow r3 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & \underline{2} & 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -3/2 & 1 \end{array}\right)$$

$$\tfrac{1}{2} r2 \rightarrow r2 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & -3 & -2 & 0 & -3/2 & 1 \end{array}\right)$$

$$\begin{array}{l} (-2) r2 + r1 \rightarrow r1 \\ 3 r2 + r3 \rightarrow r3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 0 & \underline{4} & 3/2 & -3/2 & 1 \end{array}\right)$$

$$\tfrac{1}{4} r3 \rightarrow r3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 1/2 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3/8 & -3/8 & 1/4 \end{array}\right)$$

$$\begin{array}{l} 3 \cdot r3 + r1 \rightarrow r1 \\ (-2) r3 + r2 \rightarrow r2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/8 & -5/8 & 3/4 \\ 0 & 1 & 0 & -\frac{1}{4} & 3/4 & -\frac{1}{2} \\ 0 & 0 & 1 & 3/8 & -3/8 & \frac{1}{4} \end{array}\right)$$

$I \qquad A^{-1}$

Step III: $AA^{-1} = I$.

Def A linear system $Ax=b$ is $\begin{cases} \text{homogeneous if } b=\vec{0} \\ \text{nonhomogeneous if } b \neq \vec{0}. \end{cases}$

$A \in F^{m\times n}$
$x \in F^n$
$b \in F^m$

Theorem 3.8 Let K be the set of all solutions to $Ax=\vec{0}$.
Then $K = N(L_A)$, $\dim(K) = n - \text{rank}(A)$.

Proof: $K = \{v \in F^n : Av = \vec{0}\}$
$= \{v \in F^n : L_A(v) = \vec{0}\}$
$= N(L_A)$.

Dim Thm on $L_A : F^n \to F^m$

$$\underbrace{\dim(R(L_A))}_{\text{rank}(A)} + \dim(N(L_A)) = \dim(F^n)$$

Theorem 3.9 Let K be the set of all solutions to $Ax=b$.
Let K_H -- ---- ---- ---- -- $Ax=\vec{0}$.

Then for any solution s to $Ax=b$.

$K = \{s\} + K_H = \{s+v : Av = \vec{0}\}$.

Theorem 3.10 $Ax=b$ has exactly one solution $\Leftrightarrow$ A is invertible.

Theorem 3.11 $Ax=b$ has at least one solution if and only if $\text{rank}(A) = \text{rank}(A \mid b)$

Augmented

Matrix.