MA 353 Section 001/015
Spring 2021

Homework 1

Due on Jan 28 before 1pm on gradescope.

To receive full credit, use only definition, Theorems/Corollaries and rigorous reasoning.

1. (10 pts) True of false and no need to explain why:
   (a) Every vector space has a zero vector.

   (b) A vector space may have more than one zero vector.

   (c) In a vector space V over a field F, for scalars $a, b \in F$ and a nonzero vector $x \in V$, $ax = bx \Longrightarrow a = b$.

   (d) In a vector space V over a field F, for a nonzero scalar $a \in F$ and vectors $x, y \in V$, $ax = ay \Longrightarrow x = y$.

   (e) S is a nonempty subset of a vector space V, then $span(S)$ equals the intersection of all subspaces of V containing S.

   (f) If vectors $x_1, x_2, \cdots, x_n$ are linearly independent, then $a_1x_1 + a_2x_2 + \cdots + a_nx_n = \vec{0}$ implies that all scalars $a_i = 0$.

   (g) Subsets of a linearly independent set must be linearly independent.

2. (10 pts) Let $S = \{(a, b), a, b \in \mathbb{R}\}$ and defined two operations as follows:
$$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1b_2)$$
$$c \odot (a, b) = (ca, cb), \forall c \in \mathbb{R}.$$
Discuss whether S is a vector space over $\mathbb{R}$ with the addition and scalar multiplication above. If yes, prove it. If not, explain the reason.

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4. (10 pts) For an $n \times n$ matrix A, its trace is defined as sum of all diagonal entries
$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$
Prove that all real $n \times n$ matrices having trace equal to zero form a subspace of $\mathbb{R}^{n\times n}$ (the vector space of all real $n \times n$ matrices).

5. (30 pts) Let $\mathbb{R}^+ := \{x \in \mathbb{R} | x > 0\}$ be the set of positive numbers. Define vector addition $\oplus$ and scalar multiplication $\odot$ as the following:
$$x \oplus y = xy, \quad \forall a \in \mathbb{R}, \ a \odot x = x^a.$$
Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$.

6. (10 pts) Determine whether the following sets are subspaces of $\mathbb{R}^3$ under addition and scalar operations defined in $\mathbb{R}^3$.

   (a) $W_1 = \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a = c + 2\}.$

   (b) $W_2 = \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a - 4b - c = 0\}.$

7. (10 pts) Consider the vector space
$$V = \{\text{all real-valued single variable continuous functions}\}$$
over $\mathbb{R}$. A function $f(x) \in V$ is called even if $f(-x) = f(x), \forall x \in \mathbb{R}$. A function $f(x) \in V$ is called odd if $f(-x) = -f(x), \forall x \in \mathbb{R}$. Prove that the set of all odd functions form a subspace of V.

8. (10 pts)

   (a) In $V = \mathbb{R}^{1\times 3}$, determine whether $(2, -1, 1)$ is in the span of the set $S = \{(1, 0, 2), (-1, 1, 1)\}$.

   (b) Consider the vector space consisting of all polynomials with real coefficients, determine whether $x^3 - 3x + 5$ can be expressed as a linear combination of $x^3 + 2x^2 - x + 1$ and $x^3 + 3x^2 - 1$.