MA 353 Section 001/015
Spring 2021

Homework 10

Due on **April 22nd Thursday before 1pm** on gradescope.

Recall that we only consider $F = \mathbb{R}$ or $\mathbb{C}$ whenever inner product is involved.

1. (20 pts) True or False. No need to give reasoning.
   (a) For $A \in \mathbb{C}^{n\times n}$, $AA^* = A^*A$ if and only if there exist n orthonormal eigenvectors of A.
   (b) Unitary matrices have n orthonormal eigenvectors.
   (c) Orthogonal matrices are diagonalizable.
   (d) Unitary matrices must have real eigenvalues.
   (e) For a self-adjoint operator $T : V \longrightarrow V$, its matrix representation $[T]_\beta$ under any ordered basis β is complex Hermitian.
   (f) For an orthogonal operator $T : V \longrightarrow V$, its matrix representation $[T]_\beta$ under any ordered basis β is an orthogonal matrix.
   (g) Similar matrices have the same eigenvalues.
   (h) Unitarily equivalent matrices have the same eigenvalues.
   (i) If a matrix A is unitarily equivalent to an unitary matrix, then A is also unitary.
   (j) Orthogonal operators/matrices are also normal.
2. (10 pts) Assume $Q \in \mathbb{C}^{n\times n}$ has orthonormal columns. Prove that Q also has orthonormal rows.
3. (20 pts) For a linear $T : V \longrightarrow V$ over a finite dimensional inner product space V with $F = \mathbb{C}$ and $dim(V) = n$, prove that T has n orthonormal eigenvectors with all eigenvalues $|\lambda_i| = 1$ if and only if T is unitary.
4. (10 pts) For a normal operator $T : V \longrightarrow V$, prove that its matrix representation $[T]_\beta$ under an orthonormal basis β is normal.
5. (10 pts) For
$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix},$$
find an orthogonal or unitary matrix P such that P^*AP is diagonal.

6. (20 pts) Let A and B be two unitarily equivalent $n \times n$ complex matrices.

   (a) Prove that $tr(A^*A) = tr(B^*B)$.

   (b) Use (a) to prove

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |B_{ij}|^2.$$

7. (10 pts) Let $A \in \mathbb{F}^{n \times n}$ be a real symmetric matrix or a complex normal matrix, then it has n eigenvalues in the corresponding field including repeated ones: $\lambda_1, \lambda_2, \cdots, \lambda_n$. Prove that

$$tr(A) = \sum_{i=1}^{n} \lambda_i, \quad tr(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2.$$