MA 353 Section 001/015
Spring 2021

Homework 2

Due on Feb 4th before 1pm on gradescope.

To receive full credit, use only definition, Theorems/Corollaries and rigorous reasoning.

1. (10 pts) True of false and no need to explain why:
   (a) $\{\vec{0}\}$ is a linearly independent set.

   (b) Every vector space that is generated by a finite set has a basis.

   (c) Every vector space has a unique basis.

   (d) Every subspace of a finite dimensional vector space is finite dimensional.

   (e) $\{\vec{0}\}$ can be a vector space over any field F.

   (f) V is a n-dimensional vector space. $S \subset V$ has n vectors. Then S is linearly independent if and only if S spans V.

   (g) For a mapping T from a finite dimensional vector space V (over F) to a finite dimensional vector space W (over F), if $T(x+y) = T(x) + T(y)$, then T is a linear transformation.

   (h) For a mapping T from a finite dimensional vector space V (over F) to a finite dimensional vector space W (over F), if T is a linear transformation, then $T(\vec{0}_V) = \vec{0}_W$, where $\vec{0}_V$ denotes the zero vector in V.

   (i) For a mapping T from a finite dimensional vector space V (over F) to a finite dimensional vector space W (over F), if T is not surjective, then T cannot be a linear transformation.

   (j) For two linear transformations T_1, T_2 from a finite dimensional vector space V (over F) to a finite dimensional vector space W (over F), if T_1 and T_2 are the same on a basis of V, then T_1 and T_2 are the same on V.

2. (20 pts)

   Definition 1. *Consider $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of all rational numbers. Then $\mathbb{Q}(\sqrt{2})$ is a field under conventional addition and multiplication for numbers. Under conventional addition and multiplication for numbers, $V = \mathbb{Q}(\sqrt{2})$ is also a vector space over the field $F = \mathbb{Q}(\sqrt{2})$, and the dimension is one because $\{1\}$ is a basis.*

   Now consider the following mapping from a vector space $V = \mathbb{Q}(\sqrt{2})$ to another vector space $W = \mathbb{Q}(\sqrt{2})$ over the same field $F = \mathbb{Q}(\sqrt{2})$ defined as:
   $$T(a + b\sqrt{2}) = a.$$
   Determine whether T is a linear transformation.

3. (20 pts)

   Definition 2. In a field F, if $1 + 1 + \cdots + 1 = 0$ (summation of n ones is equal to zero), the number n is called the characteristics of the field. For example, in the field $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$, $1 + 1 + 1 = mod(3, 3) = 0$, thus the characteristics of this field is 3.

   Now consider a vector space V over a field F, and assume the characteristic of F is not 2 (in other words, $1 + 1 \neq 0$ in F). Let u and v be two distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
   Hint: In a general field, we only have 0 and 1 defined, and anything else does not have a specific name, e.g., we cannot say $1 + 1 = 2$ because 2 is not defined. In F, $1 + 1 \neq 0$ implies $\frac{1}{1+1}$ exists ($\frac{1}{1+1}$ means the inverse to $1 + 1$ for multiplication). Thus in F, $c + c = 0 \Rightarrow c(1 + 1) = 0$, then multiplying both sides by $\frac{1}{1+1}$, we get $c = 0$.

4. (10 pts) Let V be a vector space over a field F with $\dim V = n$. Let $S = \{w_1, \ldots, w_m\} \subset V$ be a set of linearly independent vectors. Using Replacement Theorem (Theorem 1.10) to show that $m \leq n$. Furthermore, $m = n$ if and only if S is a basis of V.

5. (10 pts) Determine whether the set $\{1+2x+x^2, -2+3x-x^2, 1-x+6x^2\}$ is a basis for $P_2(\mathbb{R})$.

6. (10 pts) Let u, v, w be distinct vectors in V over a general field F. Assume $\{u, v, w\}$ is a basis of V. Prove that $\{u + v + w, v + w, w\}$ is also a basis.

7. (10 pts) Find a basis for the following subspace for $\mathbb{R}^5$:

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \in \mathbb{R}^5 : b = c = d \quad \text{and} \quad a + e = 0. \right\}$$

8. (10 pts) The rotation operation of rotating a vector in $x - y$ plane by an angle θ counter clockwise can be denoted as a mapping $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$:

$$T(x, y) = (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y).$$

Verify that this is a linear transformation.