MA 353 Section 001/015
Spring 2021

# Homework 3

Due on Feb 11th before 1pm on gradescope.

**To receive full credit, use only definition, Theorems/Corollaries and rigorous reasoning.**

1. (10 pts) True of false and no need to explain why:
   (a) Given any $x_1, x_2 \in V$ and any $y_1, y_2 \in W$, there exists a linear transformation $T : V \longrightarrow W$ s.t. $T(x_1) = y_1$ and $T(x_2) = y_2$.

   (b) If two linear transformations $T$ and $U$: $V \longrightarrow W$ agree on a basis of $V$, then $T$ and $U$ are the same.

   (c) For two linear transformations $U, T$ from a finite dimensional vector space $V$ (over $F$ with ordered basis $\beta$) to a finite dimensional vector space $W$ (over $F$ with ordered basis $\gamma$), $[T]_\beta^\gamma = [U]_\beta^\gamma \Longrightarrow T = U$.

   (d) For an invertible linear transformation $T$ from a finite dimensional vector space $V$ (over $F$ with ordered basis $\beta$) to a finite dimensional vector space $W$ (over $F$ with ordered basis $\gamma$), $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\beta^\gamma$.

   (e) $F^{m\times n}$ is isomorphic to $F^{m+n}$.

   (f) Let $V, W, Z$ be vector spaces with finite ordered bases $\alpha, \beta, \gamma$ and let $T : V \longrightarrow W$ and $U : W \longrightarrow Z$ be linear. Then $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$.

   (g) Let $V, W, Z$ be vector spaces with finite ordered bases $\alpha, \beta, \gamma$ and let $T : V \longrightarrow W$ be linear. Then $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$.

   (h) For a n-dimensional vector space $V$ over F, $\mathcal{L}(V)$ is a vector space over $F$ with dimension $n^2$.

   (i) $T : V \longrightarrow W$ is linear, then $T$ is invertible if and only if $T$ is one-on-one and onto.

(j) Let $V, W$ be vector spaces, then $\mathcal{L}(V, W) = \mathcal{L}(W, V)$.

(k) Let $V, W, Z$ be vector spaces of the same finite dimension, then $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(W, Z)$.

2. (10 pts) Let $T : \mathbb{R}^{1\times 2} \longrightarrow \mathbb{R}^{1\times 2}$ be linear satisfying $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. Find what $T(2,3)$ is. Is $T$ one-to-one?

3. (10 pts) Prove Theorem 2.2. for the case that $\beta$ is infinite: $\beta$ is a basis of $V$, prove that $R(T) = Span(T(\beta)) = Span\{T(v) : v \in \beta\}$ for a linear transformation $T : V \longrightarrow W$.
**Hint**: only finite sums are used in the definition of these notations: *linear combination, span, generate.* For example, the vector space of all polynomials with real coefficients $P(\mathbb{R})$ has an infinite basis $\beta = \{1, x, x^2, \cdots\}$. Then $\forall p(x) \in P(\mathbb{R})$, we have $p(x) \in Span(\beta)$, which means (by definition) the following: there exists finite vectors in $\beta$ s.t. $p(x)$ is a linear combination of them (of course a polynomial $p(x)$ has a finite degree).

4. (10 pts)

**Definition 1.** For two subsets $S_1, S_2$ of a vector space $V$, $S_1 + S_2$ is defined to be the set $\{x + y : x \in S_1, y \in S_2\}$.

**Definition 2.** If $W_1, W_2$ are two subspaces of $V$ s.t. $W_1 \cap W_2 = \{\vec{0}\}$ and $V = W_1 + W_2$, then we denote $V = W_1 \oplus W_2$ ($V$ is called direct sum of $W_1$ and $W_2$).

Now consider a linear transformation $T : V \longrightarrow V$. Assume $V$ is finite dimensional and $R(T) \cap N(T) = \{\vec{0}\}$, prove that $V = R(T) \oplus N(T)$.

5. (10 pts) Define $T : \mathbb{R}^{2\times 2} \longrightarrow P_2(\mathbb{R})$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$. Find $[T]_\beta^\gamma$.

6. (10 pts) Let $V$ be a vector space on an abstract field $F$ with ordered basis $\beta = \{v_1, \cdots, v_n\}$. Let $v_0 = \vec{0}$. By Theorem 2.6 on page 73, there exists a linear transformation $T : V \longrightarrow V$ s.t. $T(v_j) = v_j + v_{j-1}$ for

$j = 1, 2, \cdots, n$. Compute $[T]_\beta$.

7. (10 pts) Let $V$ be a vector space on an abstract field $F$ and $T : V \longrightarrow V$ is linear. Let $T_0$ denote zero transformation, i.e., sends all vector of $V$ to $\vec{0}$. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

8. (20 pts) Let $V$ be a finite dimensional vector space and $T : V \longrightarrow V$ be linear. If $rank(T) = rank(T^2)$, prove that $R(T) \cap N(T) = \{\vec{0}\}$ ( thus by Problem 4, $V = R(T) \oplus N(T)$).
   **Hint**: Restrict $T$ to $R(T)$ (a subspace of $V$) then it is also a linear transformation. Call it $\bar{T}$, then it might help to consider $\bar{T} : R(T) \longrightarrow R(T^2)$.

9. (10 pts) Let $V, W$ be finite dimensional vector spaces and $T : V \longrightarrow W$ be isomorphism. Let $V_0$ be a subspace of $V$. Prove that $T(V_0)$ is a subspace in $W$ and $dim(V_0) = dim(T(V_0))$.