MA 353 Section 001/015
Spring 2021

Homework 4

Due on Feb 18th before **3am (Be Aware of the TIME)** on gradescope.

To receive full credit, use only definition, Theorems/Corollaries and rigorous reasoning.

1. (10 pts) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. (10 pts) For the ordered bases β and β' for $P_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinate to β-coordinate, i.e., find matrix Q such that $[v]_\beta = Q[v]_{\beta'}$ for any $v \in P_2(\mathbb{R})$.

$$\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}, \quad \beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}.$$

3. (20 pts)

Definition 1. For $A, B \in F^{n\times n}$, we say A is similar to B if there is an invertible matrix Q such that $B = Q^{-1}AQ$.

Recall that trace of a $n \times n$matrix $A = (a_{ij})_{n\times n}$ is defined as $tr(A) = \sum_{i=1}^{n} a_{ii}$. For two $n \times n$ matrices A, B, prove the following:

(a) $tr(AB) = tr(BA)$.

(b) $tr(A) = tr(A^T)$.

(c) If A is similar to B, then $tr(A) = tr(B)$.

4. (10 pts) For the following linear system with coefficient matrix A, first find A^{-1} then use A^{-1} to find its solution. (No credit if not using A^{-1}).

$$\begin{cases} x_1 + 2x_2 - x_3 = 5 \\ x_1 + x_2 + x_3 = 1 \\ 2x_1 - 2x_2 + x_3 = 4 \end{cases}$$

5. (10 pts) Let W be a subspace in $\mathbf{R}^4$ consisting of all vectors having entries that sum to zero. Find a basis of W.

6. (10 pts) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.
7. (10 pts) Let $A \in \mathbb{R}^{m \times n}$ have rank m and $B \in \mathbb{R}^{n \times p}$ have rank n. Determine the rank of AB and justify your answer.
   Hint: use L_A and L_B.
8. (20 pts) Express the invertible matrix A as a product of elementary matrices:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$