MA 353 Section 001/015
Spring 2021

# Homework 6

Due on Mar 11th before 1pm on gradescope.

1. (20 pts) True of false and no need to explain why:
   (a) A linear operator $T : V \longrightarrow V$ for $n$-dimensional vector space $V$ has $n$ eigenvalues.
   (b) If a real matrix has one eigenvector, then it has infinitely many eigenvectors.
   (c) There exists a square matrix with no eigenvalues.
   (d) The sum of two eigenvectors is still an eigenvector.
   (e) A linear operator $T : V \longrightarrow V$ for infinite dimensional vector space $V$ has no eigenvalues.
   (f) Similar matrices have the same eigenvectors.
   (g) If a linear operator $T : V \longrightarrow V$ for $n$-dimensional vector space $V$ has fewer than $n$ distinct eigenvalues, then it is not diagonalizable.
   (h) Two distinct eigenvectors for the same eigenvalue are linearly dependent.
   (i) Let $E_\lambda$ denote the eigenspace for eigenvalue $\lambda$. If $\lambda_1, \lambda_2$ are two distinct eigenvalues of a linear operator $T$, then $E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$.
   (j) A linear operator $T : V \longrightarrow V$ for $n$-dimensional vector space $V$ is diagonalizable if and only if the algebraic multiplicity of each eigenvalue $\lambda$ equals the dimension of $E_\lambda$.
2. (20 pts) Let $T : V \longrightarrow W$ be a linear transformation from a finite dimensional vector space $V$ to a finite dimensional vector space $W$. Let $\beta, \beta'$ be ordered bases for $V$ and $Q = [I_V]_{\beta'}^{\beta}$ be the change of coordinate matrix, where $I_V : V \longrightarrow V$ is the identity map. Let $\gamma, \gamma'$ be ordered bases for $W$ and $P = [I_W]_{\gamma'}^{\gamma}$ be the change of coordinate matrix, where $I_W : W \longrightarrow W$ is the identity map. Prove that $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q$.
3. (10 pts) Let $A \in F^{n\times n}$ and let $\gamma = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for $F^n$. Let $Q$ be the matrix with $v_i$ being i-th column. Prove that $[L_A]_\gamma = Q^{-1}AQ$.
4. (20 pts) Determine whether there exists a real matrix $Q$ such that $Q^{-1}AQ$ is diagonal. If yes, find $Q$ and $Q^{-1}AQ$. If no, further determine

whether there exists a complex matrix $Q$ such that $Q^{-1}AQ$ is diagonal, and find both $Q$ and $Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$.

5. (10 pts) For $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$, use diagonalization to find an expression of $A^n$ for an arbitrary positive integer $n$.

6. (10 pts) For a diagonalizable matrix $A$, let $\lambda_1, \cdots, \lambda_k$ be its all eigenvalues with algebraic multiplicities $m_1, m_2, \cdots, m_k$. Prove that

$$tr(A) = \sum_{i=1}^{k} m_i \lambda_i.$$

Hint: Problem 3 in HW#4 might help.

7. (10 pts) Consider $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by

$$T(p(x)) = p(0) + p(1)(x + x^2).$$

Determine whether $T$ is diagonalizable by some basis $\beta$. If yes, find $\beta$ and $[T]_\beta$.