MA 353 Section 001/015
Spring 2021

Homework 8

Due on **April 8th Thursday before 1pm** on gradescope.

1. (10 pts) Let $V = C([-1,1])$. Let W_e and W_o denote subspaces of V consisting of the even and odd functions (see HW#1 Problem 7 for definition), respectively. Prove that $W_e^{\perp} = W_o$, where the inner product is defined as
$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$
2. (10 pts) Let V be an inner product space (not necessarily finite dimensional) and let $S = \{v_1, \cdots, v_k\}$ be an orthonormal set in V.
   (a) Prove *Bessel's Inequality*:
$$\forall x \in V, \quad \|x\|^2 \geq \sum_{i=1}^{k} |\langle x, v_i\rangle|^2$$
   (b) Prove that Bessel's inequality above is an equality if and only if $x \in Span(S)$.

   Hint: use Theorem 6.6.
3. (20 pts)
   (a) Consider $V = P(\mathbb{R})$ with inner product
$$\langle f, g\rangle = \int_{0}^{1} f(t)g(t)dt.$$
   Find the projection of $4 + 3x - 2x^2$ onto the subspace $W = P_1(\mathbb{R})$.

   (b) Consider $V = \mathbb{R}^3$ with standard inner product, find projection of $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ onto
$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 3y - 2z = 0 \right\}.$$
4. (10 pts) Let V be a finite-dimensional product space. Let $T : V \longrightarrow V$ be a linear operator. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$. **Hint**: use Theorem 6.10 and the fact that $(A^*)^{-1} = (A^{-1})^*$ for a square matrix (because $(A^T)^{-1} = (A^{-1})^T$ and $(\bar{A})^{-1} = \overline{A^{-1}}$.)

5. (10 pts) Let V be a product space (maybe not be finite-dimensional). Prove that $\|T(x)\| = \|x\|, \forall x \in V$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle, \forall x, y \in V$.
   Hint: use polar identity in HW#7.

6. (10 pts) Consider $V = P_1(\mathbb{R})$ with $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$ and the linear operator $T : V \longrightarrow V$ defined by $T[f(x)] = f'(x) + 3f(x)$. Evaluate $T^*[g(x)]$ where $g(x) = 4 - 2x$.
   Hint: To find T^*, one way is to use its matrix representation. Be ware that β must be an orthonormal basis in Theorem 6.10.

7. (10 pts) Use the least square approximation to find the best fits with (i) a linear function and (ii) a quadratic function, for the following data on x-y plane
$$\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$$

8. (20 pts) Let V be a finite-dimensional product space. Let $T : V \longrightarrow V$ be a linear operator. Prove that
   (a) $N(T^*T) = N(T)$. Deduce that $rank(T^*T) = rank(T)$.
   (b) $rank(T) = rank(T^*)$. Deduce that $rank(TT^*) = rank(T)$.