MA 353 Section 001/015
Spring 2021

Homework 9

Due on **April 15th Thursday before 1pm** on gradescope.

Recall that we only consider $F = \mathbb{R}$ or $\mathbb{C}$ whenever inner product is involved.

1. (20 pts) True or False. No need to give reasoning.
   (a) Unitary operators are also normal.
   (b) T and its adjoint have the same eigenvectors.
   (c) For $T : V \longrightarrow V$ with a finite dimensional inner product space V, then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis of V.
   (d) Normal matrices/operators are diagonalizable.
   (e) Self-adjoint matrices/operators are diagonalizable.
   (f) Orthogonal matrices/operators are diagonalizable.
   (g) If a matrix $A \in F^{n\times n}$ is normal, then so is L_A.
   (h) For a matrix $A \in F^{n\times n}$, $(L_A)^* = L_{A^*}$.
   (i) For any two linear operators and scalars a, b,:
   $$(aT + bU)^* = aT^* + bU^*.$$
   (j) For a finite dimensional inner product space V, every linear $T : V \longrightarrow V$ has an adjoint operator.

2. (20 pts) Consider a real skew-symmetric matrix $A \in \mathbb{R}^{n\times n}$, i.e., $A^T = -A$. The matrix A has n complex eigenvalues by Fundamental Theorem of Algebra. Prove that A has n purely imaginary eigenvalues (i.e., real parts of eigenvalues are zero).
   Hint: By Lemma on Page 370 (or Lecture Notes on April 8th), a real self-adjoint operator over n-dimensional vector space V has n real eigenvalues including repeated ones. This implies that a real symmetric $n \times n$ matrix B has n real eigenvalues (simply apply the Lemma to the operator L_B). Consider $T = L_A$ and mimic the proof of the Lemma.

3. (10 pts) Prove the IF direction in Theorem 6.17: for a finite-dimensional V over $F = \mathbb{R}$, if there exists an orthonormal basis β for V consisting of eigenvectors of $T : V \longrightarrow V$, then T is self-adjoint.

4. (10 pts) For a normal linear operator $T : V \longrightarrow V$ where V could be an infinite dimensional inner product space, prove that $T - cI$ is also normal with any $c \in F$.
5. (10 pts) For $T : V \longrightarrow V$ with a finite dimensional inner product space V, if γ is an orthonormal basis of V, then we have $[T^*]_\gamma = [T]_\gamma^*$. Derive the relation between two matrices $[T^*]_\beta$ and $[T]_\beta$ for any ordered basis β.
   Hint: use the change of coordinate matrix $Q = [I]_\beta^\gamma$.
6. (20 pts) Let $T : V \longrightarrow V$ be a normal operator on a finite dimensional inner product space V. Prove that
   (a) $N(T) = N(T^*)$. Hint: use HW#8.
   (b) $R(T^*)^\perp = N(T)$. Deduce $R(T) = R(T^*)$.
7. (10 pts) Let $T : V \longrightarrow V$ be a self-adjoint operator on a finite dimensional inner product space V. $F = \mathbb{C}$. Prove that $\forall x \in V$:

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$