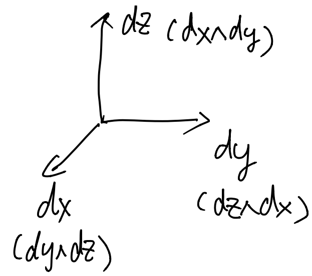
2-form $\omega = F\,dy\wedge dz + G\,dz\wedge dx + H\,dx\wedge dy$

$\sim \langle F, G, H\rangle$

$d\omega = (\nabla\cdot\langle F,G,H\rangle)\,dx\wedge dy\wedge dz$



Parametric Surface

S ① $\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$ ② $(u,v)\in D$ ③ orientation : $T_u\times T_v$ or $T_v\times T_u$

Ex: upper half of unit sphere with outward normal vector.

① $\begin{cases} x = u \\ y = v \\ z = \sqrt{1-u^2-v^2} \end{cases}$ ② $u^2+v^2\le 1$ ③ $T_u\times T_v$: outward normal.

Surface Integral : $\iint_S \omega = \iint_S F\,dy\wedge dz + G\,dz\wedge dx + H\,dx\wedge dy$

(Def) $= \iint_D \left[F\frac{\partial(y,z)}{\partial(u,v)} + G\frac{\partial(z,x)}{\partial(u,v)} + H\frac{\partial(x,y)}{\partial(u,v)}\right]du\,dv$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$dy\wedge dz = \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)\wedge\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right) = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)du\wedge dv$$

$$\begin{array}{c} \\ T_u \\ T_v \end{array}\begin{vmatrix} F & G & H \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = F\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} - G\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} + H\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$+ G\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}$$

$$\| $$

$$\langle F,G,H\rangle\cdot(T_u\times T_v)$$

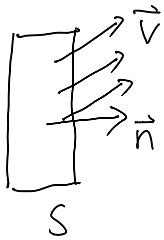
$d\vec{S} = T_u\times T_v\,du\,dv$

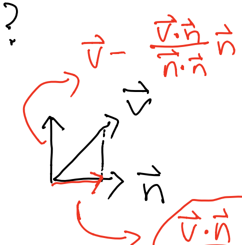
$$\iint_S \omega = \iint_D \langle F,G,H\rangle\cdot(T_u\times T_v)\,du\,dv$$

$$= \iint_S \langle F,G,H\rangle\cdot d\vec{S}$$

$$\iint_S \omega = \iint_S \langle F,G,H\rangle\cdot d\vec{S} \overset{\text{Def}}{=} \iint_D \langle F,G,H\rangle\cdot T_u\times T_v\,du\,dv$$

How much water is flowing through S in unit time if $\underbrace{\langle F, G, H\rangle}_{\vec F}$ denotes water velocity?

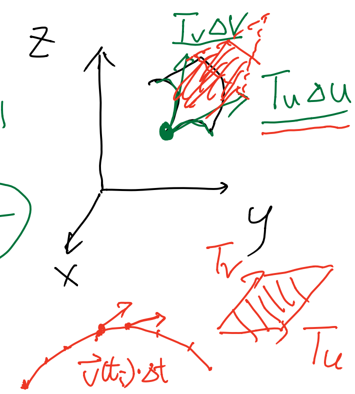


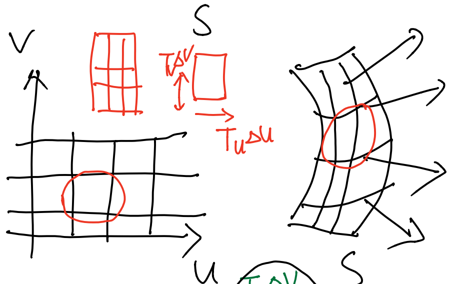


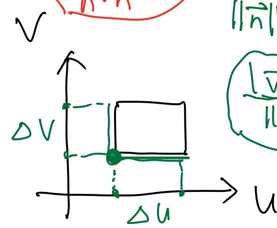
$$\frac{\vec v\cdot\vec n}{\vec n\cdot\vec n}\vec n$$

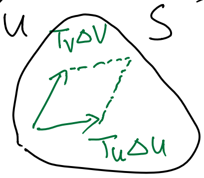
$$\left\|\frac{\vec v\cdot\vec n}{\vec n\cdot\vec n}\vec n\right\|\cdot 1\cdot \text{Area}(S)$$

$$\frac{|\vec v\cdot\vec n|}{\|\vec n\|^2}\|\vec n\| = \frac{|\vec v\cdot\vec n|}{\|\vec n\|}$$









Area of ▱ is $\|T_u\Delta u\times T_v\Delta v\| = \|T_u\times T_v\|\Delta u\Delta v$

Normal Vector to ▱ (Surface) is $T_u\times T_v$

The normal velocity of water is $\dfrac{\vec F\cdot(T_u\times T_v)}{\|T_u\times T_v\|}$

$$\lim_{\substack{\Delta u\to 0\\ \Delta v\to 0}} \sum_{\substack{\text{all small}\\ \text{surfaces}}} \frac{\vec F\cdot(T_u\times T_v)}{\|T_u\times T_v\|}\cdot\|T_u\times T_v\|\Delta u\Delta v = \iint_D \vec F\cdot(T_u\times T_v)\,du\,dv$$

Ex: $S\begin{cases} x=u\\ y=v\\ z=\sqrt{1-u^2-v^2}\end{cases}$ $\qquad T_u\times T_v$

$$\iint_S dz\wedge dx = \iint_D \langle 0,1,0\rangle\cdot T_u\times T_v\,du\,dv$$

$$\begin{vmatrix} 0 & 1 & 0\\ 1 & 0 & \frac{-2u}{2\sqrt{1-u^2-v^2}}\\ 0 & 1 & \frac{-2v}{2\sqrt{1-u^2-v^2}}\end{vmatrix} = -1\cdot\begin{vmatrix} 1 & \frac{-2u}{2\sqrt{1-u^2-v^2}}\\ 0 & \frac{-2v}{2\sqrt{1-u^2-v^2}}\end{vmatrix} = \frac{v}{\sqrt{1-u^2-v^2}}$$

$$\iint_S dz\wedge dx = \iint_D \frac{v}{\sqrt{1+u^2+v^2}}\,\underset{r\,dr\,d\theta}{\underline{du\,dv}} \qquad \begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$= \int_0^{2\pi}\int_0^1 \frac{r\sin\theta}{\sqrt{1+r^2}}\, r\, dr\, d\theta$$

$$= \underline{\left[\int_0^{2\pi}\sin\theta\, d\theta\right]}\cdot \underline{\int_0^1 \frac{r^2}{\sqrt{1+r^2}}\, dr}$$

$$= 0.$$