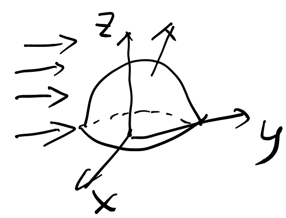
Ex: $\iint_S dz\wedge dx = 0$ $\qquad$ $S: \begin{cases} x = u \\ y = v \\ z = \sqrt{1-u^2-v^2} \end{cases}$ $\qquad T_u \times T_v$



$\langle 0,1,0\rangle$

$0\cdot dy\wedge dz + 1\cdot dz\wedge dx + 0\cdot dx\wedge dy$

Theorem : $\iint_S \omega$ is independent of parametrization of S.

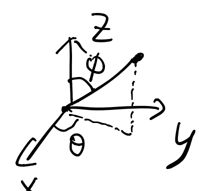
Remark: $-S$ denotes the oppositie orientation of S

$$\iint_{-S} \omega = -\iint_S \omega$$

$\iint_S \omega = \iint_D \vec{F}\cdot(T_u\times T_v)\,du\,dv$

$= -\iint_D \vec{F}\cdot(T_v\times T_u)\,du\,dv = -\iint_{-S}\omega$

Ex: $\iint_S dz\wedge dx$ where S is upper half of unit sphere with upward normal.



Sol: $\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases}$ $\qquad T_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$

$\qquad T_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$

$T_\phi \times T_\theta$ $\qquad 0\le\phi\le\frac{\pi}{2},\ 0\le\theta\le\pi$

$$= \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = i\begin{vmatrix} \cos\phi\sin\theta & -\sin\phi \\ \sin\phi\cos\theta & 0 \end{vmatrix} - j\begin{vmatrix} \cos\phi\cos\theta & -\sin\phi \\ -\sin\phi\sin\theta & 0 \end{vmatrix}$$

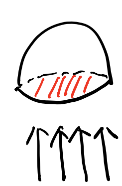
$$+ k\begin{vmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta \\ -\sin\phi\sin\theta & \sin\phi\cos\theta \end{vmatrix} = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \underbrace{\cos\phi\sin\phi}_{\frac{1}{2}\sin 2\phi \,\ge 0}\rangle$$

$$\iint_S dz\wedge dx = \iint_D \langle 0,1,0\rangle\cdot T_\phi\times T_\theta\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\frac{\pi}{2}} \sin^2\phi\sin\theta\, d\phi\right]d\theta = \left[\int_0^{\frac{\pi}{2}}\sin^2\phi\, d\phi\right]\left[\int_0^{2\pi}\sin\theta\, d\theta\right]$$

$= 0.$

Ex: Same S $\qquad \iint_S dx\wedge dy = \iint_D \langle 0,0,1\rangle\cdot T_\phi\times T_\theta\, d\phi\, d\theta$

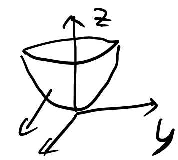


$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2\phi\, d\phi\, d\theta = 2\pi\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\phi\, d\phi$$

$$= 2\pi \left.\frac{-\cos 2\phi}{4}\right|_0^{\frac{\pi}{2}} = \frac{\pi}{2}\left[-\cos\pi - (-\cos 0)\right] = \pi$$

$d(dx\wedge dy)=0$

Ex: S: Paraboloid $z=x^2+y^2$ below $z=1$ with downward normal



$$\omega = z\,dx\wedge dy + \frac{x}{2}\,dy\wedge dz \sim \langle \tfrac{x}{2}, 0, z\rangle$$

$$\iint_S \omega$$

Sol: $S: \begin{cases} x = u \\ y = v \\ z = u^2+v^2 \end{cases}$, $(u,v)\in$ unit disk, $T_u = \langle 1, 0, 2u\rangle$, $T_v = \langle 0, 1, 2v\rangle$

$$T_u\times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = i(-2u) - j\cdot 2v + k\cdot 1 = \langle -2u, -2v, 1\rangle$$

$T_v\times T_u = \langle 2u, 2v, -1\rangle$ is downward normal.

$$\iint_S \omega = \iint_D \langle \tfrac{x}{2}, 0, z\rangle\cdot T_v\times T_u \, dv\,du$$

$$= \iint_D \underline{\langle \tfrac{u}{2}, 0, u^2+v^2\rangle \cdot \langle 2u, 2v, -1\rangle}\, dv\,du$$

$$= \iint_D -v^2\, dv\,du \qquad \begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$= \int_0^{2\pi}\int_0^1 -r^2\sin^2\theta\, r\,dr\,d\theta$$

$$= \int_0^1 -r^3\,dr \cdot \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= -\int_0^1 r^3\,dr \cdot \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\left[\frac{r^4}{4}\Big|_0^1\right]\cdot\left[\int_0^{2\pi} \frac{\theta - \frac{1}{2}\sin 2\theta}{2}\, d\theta\right]$$

$$= -\frac{1}{4}\cdot\pi = -\frac{\pi}{4}$$