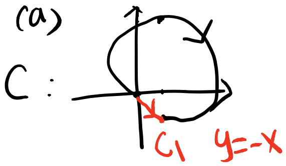
4. $\int_C \sqrt{e^{\cos^3(x+y)}}\,(dx+dy)$

(a)



$\alpha = \sqrt{e^{\cos^3(x+y)}}\,dx + \sqrt{e^{\cos^3(x+y)}}\,dy$

$F(x,y)\,dx + G(x,y)\,dy$

$F_y = G_x \Rightarrow \alpha$ is closed $\Rightarrow \alpha$ is exact $\Rightarrow \alpha = df$

$\alpha = F\,dx + G\,dy + H\,dz$

$\nabla \times \langle F, G, H \rangle = \vec{0} \Rightarrow \alpha$ is closed

$\alpha = F(x,y)\,dx + G(x,y)\,dy + 0\cdot dz$

$\nabla \times \langle F(x,y), G(x,y), 0 \rangle = \langle 0, 0, G_x - F_y \rangle$

$\langle x(t_0), y(t_0), z(t_0) \rangle$

Theorem 1-form α is exact ($\alpha = df$), a curve C starts at $\vec{x}(t_0)$ and ends at $\vec{x}(t_1)$, then

① $\int_C \alpha = \int_C df = f(\vec{x}(t_1)) - f(\vec{x}(t_0))$

② For a loop ($\vec{x}(t_1) = \vec{x}(t_0)$) C, $\oint_C \alpha = 0$

③ $\int_C \alpha$ is path-independent

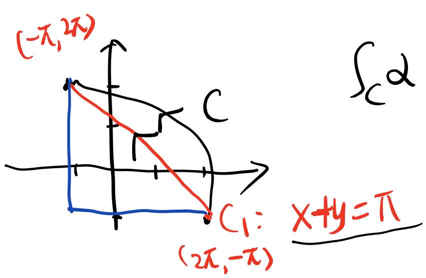
$\int_C \alpha = \int_{C_1} \alpha$

$C_1: \begin{cases} x = t \\ y = -t \end{cases} \quad 0 \le t \le 1.$

α is exact $\Rightarrow \int_C \alpha = \int_{C_1} \alpha = \int_0^1 \sqrt{e^{\cos^3(t-t)}}\cdot \underbrace{x'(t)}_{1}\,dt + \sqrt{e^{\cos^3(t-t)}}\,\underbrace{y'(t)}_{(-1)}\,dt$

$= 0$

(b)



$\int_C \alpha = \int_{C_1} \alpha = \int_{-\pi}^{2\pi} \sqrt{e^{\cos^3(\pi-t+t)}}\,\underbrace{x'(t)}_{(-1)}\,dt + \sqrt{e^{\cos^3(\pi-t+t)}}\,\underbrace{y'(t)}_{1}\,dt$

$C_1 \begin{cases} x = \pi - t \\ y = t \end{cases}, \quad -\pi \le t \le 2\pi$

5. $\int_C \alpha \qquad \alpha = F(x,y)\,dx + G(x,y)\,dy = \frac{(x+y)}{x^2+y^2}\,dx + \frac{y-x}{x^2+y^2}\,dy$

$F_y = \frac{1\cdot(x^2+y^2) - (x+y)\cdot(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2xy-2y^2}{(x^2+y^2)^2} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$

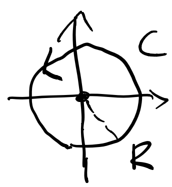
$$G_x = \frac{(-1)\cdot(x^2+y^2)-(y-x)\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2x^2}{(x^2+y^2)^2} = \frac{+x^2-2xy-y^2}{(x^2+y^2)^2}$$

$F_y = G_x \Rightarrow \alpha$ is closed $\not\Rightarrow$ α is exact

because F, G are not C^1

α is exact on $\begin{cases} \text{the first quadrant} \\ \text{the upper half of the plane} \end{cases}$

(a)



$$C: \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \quad 0 \le t \le 2\pi$$

$$F\,dx = F\,x'(t)\,dt = \frac{R\cos t + R\sin t}{R^2}(-R\sin t)\,dt$$

$$= [-\sin t\cos t - \sin^2 t]\,dt$$

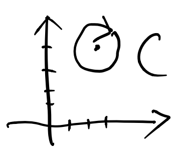
$$G\,dy = G\,y'(t)\,dt = \frac{R\sin t - R\cos t}{R^2}R\cos t\,dt$$

$$= [\sin t\cos t - \cos^2 t]\,dt$$

$$\alpha = F\,dx + G\,dy = [-\sin^2 t - \cos^2 t]\,dt = -1\cdot dt$$

$$\int_C \alpha = \int_0^{2\pi}(-1)\,dt = -2\pi$$

(b)



α is exact in $\underbrace{\text{a region}}_{\text{first quadrant}}$ containing C

$$\Rightarrow \oint_C \alpha = 0.$$

6. $C: \begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \end{cases} \quad 0 \le t \le 2\pi \qquad \int_C ds$

$$ds = \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt = \sqrt{[-\sin t + \sin t + t\cos t]^2 + [\cos t - \cos t + t\sin t]^2}\,dt$$

$$= \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = \sqrt{t^2}\,dt = |t|\,dt = t\,dt$$

$$\int_C ds = \int_0^{2\pi} t\,dt = \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2 .$$

7. $f(x,y,z) = 3x^2y - y^3 + z$

$$df = 6xy\,dx + (3x^2 - 3y^2)\,dy + 1 \cdot dz$$

$$d(df) = 0 \qquad df \wedge df = 0$$

8. $C: \begin{cases} x = t \\ y = t^2 \\ z = t^4 \end{cases}, \quad 0 \le t \le 1$

$$\alpha = yz e^{xz} dx + e^{xz} dy + xy e^{xz} dz$$

$$d\alpha$$

$$\nabla \times \langle yz e^{xz}, e^{xz}, xy e^{xz} \rangle$$

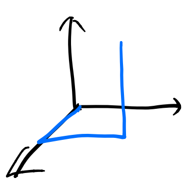
$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz e^{xz} & e^{xz} & xy e^{xz} \end{vmatrix}$$

$$= i \begin{vmatrix} \partial_y & \partial_z \\ e^{xz} & xy e^{xz} \end{vmatrix} - j \begin{vmatrix} \partial_x & \partial_z \\ yz e^{xz} & xy e^{xz} \end{vmatrix} + k \begin{vmatrix} \partial_x & \partial_y \\ yz e^{xz} & e^{xz} \end{vmatrix}$$

$$= i\,(x e^{xz} - x e^{xz}) - j\,(x e^{xz} - y e^{xz})$$
$$+ k\,(z e^{xz} - z e^{xz}) = \vec{0}$$

$d\alpha = 0 \Rightarrow \alpha$ is closed $\Rightarrow \alpha$ is exact

$\Rightarrow \int_C \alpha$ is path-independent



$$f(x,y,z) = \int_0^x F(t,0,0)\,dt + \int_0^y G(x,t,0)\,dt$$
$$+ \int_0^z H(x,y,t)\,dt$$

$$= \int_0^x \underline{0 \cdot 0 \cdot e^{t \cdot 0}}\,dt + \underbrace{\int_0^y \underline{e^{x \cdot 0}}\,dt}_{} + \int_0^z xy e^{xt}\,dt$$

$$= y + y e^{xt}\Big|_0^z = y + y e^{xz} - y = \underline{y e^{xz}}$$

$$\left(\frac{\partial}{\partial t} y e^{xt} = xy e^{xt} \right)$$