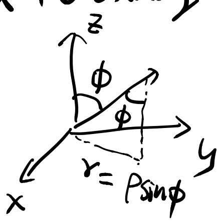
8. $\nabla(ye^{xz}) = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$

$$\int_C \alpha = \int_C df = f(1,1,1) - f(0,0,0) = e - 0 = e$$

9. $\iint_S \alpha \wedge \gamma$

$$\alpha\wedge\gamma = (x\,dx + y\,dy + z\,dz)\wedge xy\,dz$$
$$= x^2y\,dx\wedge dz + xy^2\,dy\wedge dz$$
$$= xy^2\,dy\wedge dz - x^2y\,dz\wedge dx + 0\cdot dx\wedge dy \quad \sim \underbrace{\langle xy^2, -x^2y, 0\rangle}_{\vec{F}}$$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$



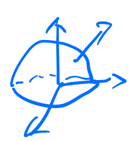
$$\boxed{\vec{F} = \langle \sin^3\phi\cos\theta\sin^2\theta,\ -\sin^3\phi\cos^2\theta\sin\theta,\ 0\rangle}$$

$$S: \begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \qquad 0\le\phi\le\pi,\ 0\le\theta\le 2\pi$$

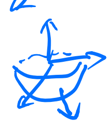
$$\begin{array}{c|ccc|} & i & j & k \\ T_\phi = & \langle\cos\phi\cos\theta, & \cos\phi\sin\theta, & -\sin\phi\rangle \\ T_\theta = & \langle -\sin\phi\sin\theta, & \sin\phi\cos\theta, & 0\ \rangle \end{array}$$

$$T_\phi\times T_\theta = i\begin{vmatrix}\cos\phi\cdot\sin\theta & -\sin\phi \\ \sin\phi\cos\theta & 0\end{vmatrix} - j\begin{vmatrix}\cos\phi\cos\theta & -\sin\phi \\ -\sin\phi\sin\theta & 0\end{vmatrix}$$
$$+ k\begin{vmatrix}\cos\phi\cos\theta & \cos\phi\sin\theta \\ -\sin\phi\sin\theta & \sin\phi\cos\theta\end{vmatrix}$$

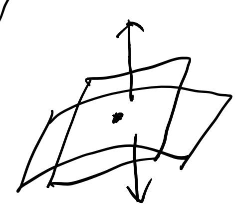
$$= \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \underbrace{\cos\phi\sin\phi}_{\frac{1}{2}\sin2\phi}\rangle$$



$0\le\phi\le\frac{\pi}{2} \Rightarrow \frac{1}{2}\sin2\phi \ge 0$



$\frac{\pi}{2}\le\phi\le\pi \Rightarrow \frac{1}{2}\sin2\phi\le 0$



$\Rightarrow \underline{T_\phi\times T_\theta}$ is outward

$\Rightarrow \underline{T_\theta\times T_\phi}$ is inward

$$\iint_S \alpha \wedge \gamma = \int_0^{\pi}\int_0^{2\pi} \underline{\vec{F} \cdot T_\theta \times T_\phi}\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi} (-\sin\phi)\left[\sin^4\phi \cos^2\theta \sin^2\theta - \sin^4\phi \cos^2\theta \sin^2\theta\right] d\theta\, d\phi$$

$$\vec{F} = \langle \sin^3\phi \cos\theta \sin^2\theta,\ -\sin^3\phi \cos^2\theta \sin\theta,\ 0 \rangle$$

$$T_\theta \times T_\phi = (-\sin\phi)\langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle$$

$$= \int_0^{\pi}\int_0^{2\pi} \underline{(-\sin\phi)\cdot 0}\, d\theta\, d\phi = 0$$

10. (a) $\omega = y\,dy \wedge dz - z\,dz \wedge dx - dx \wedge dy$

Step 0: $d\omega = (\nabla \cdot \underbrace{\langle y, -z, -1 \rangle}_{\langle y,-z,-1\rangle})\, dx \wedge dy \wedge dz = 0$

$\Rightarrow \omega$ is closed

$\Rightarrow \omega$ is exact

Step I: $\omega = \underbrace{-dx \wedge dy}_{\omega_1} + \underbrace{y\,dy \wedge dz + z\,dx \wedge dz}_{\omega_2}$

$$\beta = \left[\int_0^z y\, dt\right] dy + \left[\int_0^z t\, dt\right] dx$$

$$= \underline{(yz)\,dy + \frac{z^2}{2}\,dx}$$

$$d\beta = d(yz) \wedge dy + d\left(\frac{z^2}{2}\right) \wedge dx$$

$$\left[\begin{aligned} d[f\alpha] &= df \wedge \alpha + f\,d\alpha \\ d[f\,dx] &= df \wedge dx \end{aligned}\right]$$

$$= (0 \cdot dx + z\,dy + y\,dz) \wedge dy$$
$$+ (0 \cdot dx + 0 \cdot dy + z\,dz) \wedge dx$$
$$= y\,dz \wedge dy + z\,dz \wedge dx$$

$\omega + d\beta = -dx \wedge dy$ (no z ; no dz)

Step II: $\omega + d\beta = -dx \wedge dy$

$$\gamma = \left[\int_0^y (-1)\, dt\right] dx = -y\, dx$$

$$d\gamma = -dy \wedge dx$$

$$\omega + d\beta + d\gamma = 0$$

$$\Rightarrow \omega = d(-\beta - \gamma)$$

$$\Rightarrow \alpha = -\beta - \gamma = -yz\, dy - \frac{z^2}{2} dx + y\, dx$$

$$= \left(-\frac{z^2}{2} + y\right) dx - yz\, dy$$

$$\sim \langle -\frac{z^2}{2} + y, -yz, 0 \rangle$$

$$d^2 = 0 \begin{cases} \nabla \times (\nabla f) = \vec{0} \Rightarrow \omega = d\alpha = d(\alpha + df) \\ \nabla \cdot (\nabla \times f) = 0 \end{cases}$$

$$\nabla \times \langle -\frac{z^2}{2} + y, -yz, 0 \rangle = \nabla \times \left(\langle -\frac{z^2}{2} + y, -yz, 0 \rangle + \langle f_x, f_y, f_z \rangle \right)$$

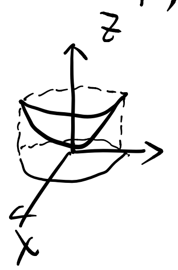
Last step: $\nabla \times \langle -\frac{z^2}{2} + y, -yz, 0 \rangle$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -\frac{z^2}{2} + y & -yz & 0 \end{vmatrix}$$

$$= i \begin{vmatrix} \partial_y & \partial_z \\ -yz & 0 \end{vmatrix} - j \begin{vmatrix} \partial_x & \partial_z \\ -\frac{z^2}{2} + y & 0 \end{vmatrix} + k \begin{vmatrix} \partial_x & \partial_y \\ -\frac{z^2}{2} + y & -yz \end{vmatrix}$$

$$= \langle +y, -z, -1 \rangle$$

11.



$z = x^2 + y^2$

$0 \le z \le 1$

$$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases}, \quad \begin{matrix} 0 \le r^2 \le 1 \\ \Downarrow \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$\begin{array}{rccc} & i & j & k \\ T_r = & \langle \cos\theta, & \sin\theta, & 2r \rangle \\ T_\theta = & \langle -r\sin\theta, & r\cos\theta, & 0 \rangle \end{array}$$

$$T_r \times T_\theta = i \begin{vmatrix} \sin\theta & 2r \\ r\cos\theta & 0 \end{vmatrix} - j \begin{vmatrix} \cos\theta & 2r \\ -r\sin\theta & 0 \end{vmatrix} + k \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle = r\langle -2r\cos\theta, 2r\sin\theta, 1\rangle$$

$$\|T_r \times T_\theta\| = r\sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}$$

$$= r\sqrt{4r^2+1} = \sqrt{4r^2+1}\,\underline{r}$$

$$\iint_S dS = \iint_0^{2\pi}\!\!{}_0^1 \underline{\|T_r \times T_\theta\|\,dr\,d\theta}$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \sqrt{4r^2+1}\,\underbrace{r\,dr}_{d\left(\frac{r^2}{2}\right)}$$

$$= \pi \int_0^1 \sqrt{4r^2+1}\, d(r^2)$$

$$= \pi \int_0^1 \sqrt{4u+1}\,\underline{du}$$

$$= \pi \int_0^1 \sqrt{4u+1}\,\frac{1}{4} d(4u+1)$$

$$= \frac{\pi}{4}\int_1^5 \sqrt{v}\,dv$$

$$= \frac{\pi}{4}\left[\frac{2}{3}v^{\frac{3}{2}}\Big|_1^5\right]$$