$$+\rho^2\sin^3\phi\sin^2\theta\, d\theta\wedge d\rho\wedge d\phi - \rho^2\cos^2\phi\sin\phi\sin^2\theta\, d\theta\wedge d\rho\wedge d\phi$$

$$= \rho^2\sin^3\phi\cos^2\theta\, d\rho\wedge d\phi\wedge d\theta + \rho^2\cos^2\phi\sin\phi\cos^2\theta\, d\rho\wedge d\phi\wedge d\theta$$

$$+\rho^2\sin^3\phi\sin^2\theta\, d\rho\wedge d\phi\wedge d\theta + \rho^2\cos^2\phi\sin\phi\sin^2\theta\, d\rho\wedge d\phi\wedge d\theta$$

$$= \rho^2\sin^3\phi\, d\rho\wedge d\phi\wedge d\theta + \rho^2\cos^2\phi\sin\phi\, d\rho\wedge d\phi\wedge d\theta$$

$$= \rho^2\sin\phi\left[\sin^2\phi+\cos^2\phi\right] d\rho\wedge d\phi\wedge d\theta$$

$$= \rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta$$

$$\iiint_V 1\cdot dx\,dy\,dz \left(= \iiint_V \rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta\right) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

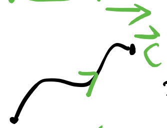
$V: 0\le\rho\le R,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi$

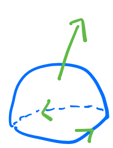
Stokes-Cartan Theorem

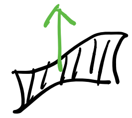
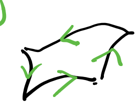
Ω is an n-dimensional manifold
(a general name for "surface")

$\partial\Omega$ denotes the boundary of Ω and it is $(n-1)$-dimensional.
Ω and $\partial\Omega$ have matching orientations.

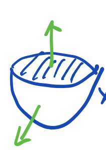
Ex: ① Ω is an interval $[a,b]$, $\partial\Omega$ are two end points.
1D $\qquad$ 0D $\quad b, -a$

② Ω is a curve , $\partial\Omega$ are two end points.
$\vec{c}(t_1)$, $\vec{c}(t_0)$ $\qquad \vec{c}(t_1), -\vec{c}(t_0)$



③ Ω is a surface , $\partial\Omega$ is a curve 
2D $\qquad$ Right hand rule $\qquad$ 1D

④ Ω is the unit solid ball, $\partial\Omega$ is the unit sphere.
3D $\qquad$ outward normal for the closed surface $\qquad$ 2D


$x^2+y^2\le z\le 1$

⑤ Ω is defined as $\{(x,y,z,t): x^2+y^2+z^2+t^2\le 1\}$,
4D ball

then $\partial\Omega$ is 3-dimensional.

Suppose ω is an $\underset{\smile}{(n-1)}$-form, then $d\omega$ is $\underline{n\text{-form}}$.

Stokes–Cartan Theorem $\displaystyle\int_{\partial\Omega}\omega = \int_{\Omega} d\omega$ with matching orientation for $\partial\Omega$ and Ω.

① $\omega = F(x)$, $\Omega = [a,b]$, $\partial\Omega = \{b, -a\}$

$$\int_{\partial\Omega} F(x) = \int_{\Omega} dF$$

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Fundamental Theorem of Calculus $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

② $\omega = f(x,y,z)$, $d\omega = f_x\,dx + f_y\,dy + f_z\,dz$

Ω : a curve $\vec{c}(t)$, $t_0 \le t \le t_1$

$\partial\Omega$: $\vec{c}(t_1)$, $-\vec{c}(t_0)$

$$\int_{\partial\Omega} f = \int_{\Omega} df$$

$$f(\vec{c}(t_1)) - f(\vec{c}(t_0)) = \int_{\vec{c}} f_x\,dx + f_y\,dy + f_z\,dz$$

Theorem (Line Integral of an exact 1-form)

$$\int_{\vec{c}} f_x\,dx + f_y\,dy + f_z\,dz = f(\vec{c}(t_1)) - f(\vec{c}(t_0))$$

③ (a) $\omega = f\,dx + g\,dy + h\,dz$ $\qquad \vec{F} = \langle F, G, H\rangle = \nabla \times \langle f, g, h\rangle$

$d\omega = F\,dy\wedge dz + G\,dz\wedge dx + H\,dx\wedge dy$

$\left.\begin{array}{l}\Omega : \text{a surface } S \\ \partial\Omega : \text{a curve } \vec{c}\end{array}\right\}$ matching orientation

$$\int_{\partial\Omega}\omega = \int_{\Omega} d\omega$$

$$\int_{\vec{c}} f\,dx + g\,dy + h\,dz = \iint_S F\,dy\wedge dz + G\,dz\wedge dx + H\,dx\wedge dy$$

$d\vec{s} = \langle dx, dy, dz\rangle$ $\qquad = \displaystyle\iint_S \langle F, G, H\rangle \cdot d\vec{S}$

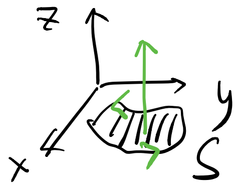
$\qquad = \displaystyle\iint_S \langle F, G, H\rangle \cdot \vec{n}\,dS$

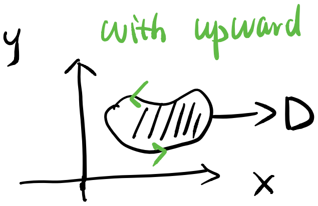
$$\begin{pmatrix} d\vec{S} = T_u \times T_v\,du\,dv \\ dS = \|T_u \times T_v\|\,du\,dv \\ \vec{n} = \dfrac{T_u \times T_v}{\|T_u \times T_v\|} \end{pmatrix}$$

Stokes Theorem

$$\iint_S (\nabla \times \langle f,g,h\rangle)\cdot \vec{n}\, dS = \int_{\vec{C}} \langle f,g,h\rangle \cdot d\vec{s}$$

(b) Consider a flat horizontal surface on x-y plane with upward normal.





$$S: \begin{cases} x = u \\ y = v \\ z = 0 \end{cases}, \quad (u,v)\in D$$

$T_u = \langle 1,0,0\rangle \quad T_v = \langle 0,1,0\rangle \quad T_u\times T_v = \langle 0,0,1\rangle$

$\|T_u\times T_v\| = 1$

$\vec{n} = \langle 0,0,1\rangle$

$\langle f,g,h\rangle = \langle P(x,y), Q(x,y), 0\rangle$

$\Rightarrow \nabla\times\langle f,g,h\rangle = \langle 0, 0, Q_y - P_x\rangle$

$$\iint_S \langle 0,0,Q_y-P_x\rangle\cdot\langle 0,0,1\rangle\, dS = \int_{\vec{C}} \langle P,Q,0\rangle\cdot\langle dx,dy,dz\rangle$$

$$= \iint_S (Q_y - P_x)\, dS$$

$$= \iint_D [Q_y(u,v) - P_x(u,v)]\, du\,dv$$

$$= \iint_D [Q_y(x,y) - P_x(x,y)]\, dx\,dy$$

Greens Theorem $\displaystyle \iint_D [Q_y - P_x]\, dx\,dy = \int_{\vec{C}} P\,dx + Q\,dy$

④ $n=3$

$\omega = F\, dy\wedge dz + G\, dz\wedge dx + H\, dx\wedge dy$

$d\omega = (\nabla\cdot\langle F,G,H\rangle)\, dx\wedge dy\wedge dz$

Ω: a 3D solid region V

$\partial\Omega$: its boundary surface $\hat{S}$ with outward normal.

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

$$\iint_S \omega \qquad \iiint_V (\nabla\cdot\langle F,G,H\rangle)\, dx\wedge dy\wedge dz$$

Gauss / Divergence Theorem

$$\iiint_V (\nabla\cdot\langle F,G,H\rangle)\,dxdydz = \iint_S \langle F,G,H\rangle\cdot\vec{n}\,dS$$

$$\int_\Omega d\omega = \int_{\partial\Omega}\omega$$

① $\int_a^b \frac{d}{dx}F(x)\,dx = F(b)-F(a)$

② $\int_{\vec{c}} \nabla f\cdot d\vec{s} = f(\vec{c}(t_1)) - f(\vec{c}(t_0))$

③ $\iint_S \nabla\times\langle f,g,h\rangle\cdot\vec{n}\,dS = \int_{\vec{c}}\langle f,g,h\rangle\cdot d\vec{s}$

$\iint_D [Q_y - P_x]\,dxdy = \int_{\vec{c}} Pdx+Qdy$

④ $\iiint_V \nabla\cdot\langle F,G,H\rangle\,dxdydz = \iint_S\langle F,G,H\rangle\cdot\vec{n}\,dS$

How to visualize a 4D ball

$$x^2+y^2+z^2+t^2\le 1$$

Look at sections for fixed t : $x^2+y^2+z^2 \le 1-t^2$

$t=-1$: $x^2+y^2+z^2=0$

$t=-\frac{1}{2}$: $x^2+y^2+z^2=\frac{3}{4}$

$t=0$: $x^2+y^2+z^2=1$

$t=\frac{1}{2}$: $x^2+y^2+z^2=\frac{3}{4}$

$t=1$: $x^2+y^2+z^2=0$