Curve : $\vec{c}(t) = \langle x(t), y(t), z(t) \rangle$ , $t_0 \le t \le t_1$

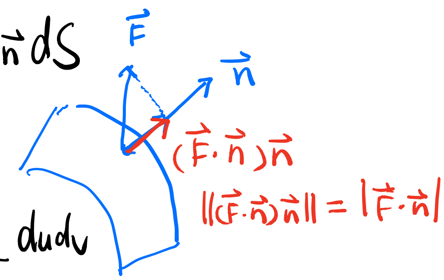
Type I $\int_C f(x,y,z)\, \underline{ds} = \int_{t_0}^{t_1} f(x(t), y(t), z(t)) \underline{\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt}$

Type II $\int_C \vec{F} \cdot \underline{d\vec{s}} = \int_C \langle F, G, H \rangle \cdot \underline{\langle dx, dy, dz \rangle} = \int_C F dx + G dy + H dz$

$= \int_{t_0}^{t_1} [F \cdot x'(t) + G \cdot y'(t) + H \cdot z'(t)] dt$

Surface : $S \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ , $(u,v) \in D$ , orientation $T_u \times T_v$

$$\left.\begin{aligned} d\vec{S} &= T_u \times T_v \, du dv \\ dS &= \|T_u \times T_v\| du dv \\ \vec{n} &= \frac{T_u \times T_v}{\|T_u \times T_v\|} du dv \end{aligned}\right\} \quad d\vec{S} = \vec{n}\, dS$$




Type I $\iint_S f dS = \iint_D \underline{f \|T_u \times T_v\|} du dv$

Type II $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \boxed{\underline{\vec{F} \cdot \vec{n}}} dS = \iint_S F dy \wedge dz + G dz \wedge dx + H dx \wedge dy$

$= \iint_D [\langle F, G, H \rangle \cdot (T_u \times T_v)] du dv$

$$\iiint_V f\, dx \wedge dy \wedge dz \overset{\text{definition}}{=\!=} \iiint_V f\, dx dy dz$$

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Stokes – Cartan Theorem $\int_\Omega \underline{d\omega} = \int_{\partial\Omega} \omega$

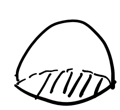
$$\iint_S d(f dx + g dy + h dz) = \oint_C f dx + g dy + h dz$$

$$\iiint_V d(f dy \wedge dz + g dz \wedge dx + h dx \wedge dy) = \oiint_S f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$$

① definition of $d$ ② orientation ③ How to use Theorem

①

Stokes Theorem $\iint_S \underline{\nabla \times} \langle f, g, h \rangle \cdot \vec{n}\, dS = \oint_C \langle f, g, h \rangle \cdot d\vec{s}$

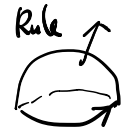


Green's Theorem $\iint_D \underline{\nabla \times} \langle P, Q, 0 \rangle \cdot \langle 0, 0, 1 \rangle dS = \oint_C \langle P, Q, 0 \rangle \cdot d\vec{s}$

$$\iint_D [Q_x - P_y]\, dx\,dy = \oint_C P\,dx + Q\,dy$$

Gauss Theorem $\displaystyle\iiint_V \nabla\cdot\langle f,g,h\rangle\, dx\,dy\,dz = \oiint_S \langle f,g,h\rangle\cdot\vec{n}\, dS$ always outward

Right Hand Rule



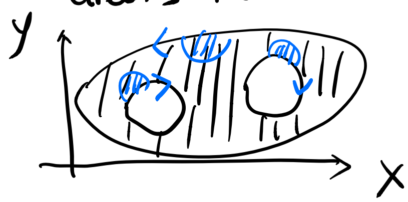
② orientation is a local property

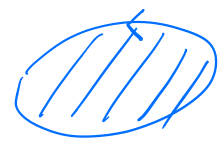
Example : Stokes Theorem on a cylinder

$$S\begin{cases} x^2+y^2=1 \\ 0\le z\le 1 \\ \text{outward normal}\end{cases}$$

Apply Right Hand Rule to a small piece of the surface.

Example: Green's Theorem





③ How to use theorems: 1) decrease the dimension for exact forms
2) increase the dimension to simplify integrand.

Ex 1 : $S$ is unit sphere with inward normal

$$\iint_S z\,dy\wedge dz - y\,dz\wedge dx + z\,dx\wedge dy$$

Sol: Let $V$ be the 3D region enclosed by $S$.

Gauss Theorem $\Rightarrow \displaystyle\iint_S \omega = -\iint_{-S}\omega = -\iiint_V d\omega = -\iiint_V \nabla\cdot\langle z,-y,z\rangle\, dx\,dy\,dz$

$$= \iiint_V 0\, dx\,dy\,dz = 0.$$

Ex 2 : $S$ is upper half of unit sphere with outward normal.

$$\iint_S z\,dy\wedge dz - y\,dz\wedge dx + z\,dx\wedge dy$$

$\nabla\cdot\langle z,-y,z\rangle = 0 \Rightarrow \omega$ is closed/exact.

Poincare's Lemma: $\omega = \underbrace{z\,dx\wedge dy}_{\omega_1} + \underbrace{y\,dx\wedge dz + z\,dy\wedge dz}_{\omega_2}$

$$\beta = \left[\int_0^z y\,dt\right]dx + \left[\int_0^z t\,dt\right]dy$$

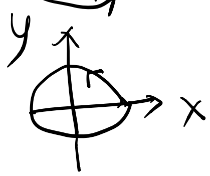
$$= yz\,dx + \frac{z^2}{2}\,dy$$

$$d\beta = y\,dz\wedge dx + z\,dy\wedge dx + z\,dz\wedge dy$$

$$\omega + d\beta = (z-z)\,dx\wedge dy = 0$$

$$\Rightarrow \omega = -d\beta = d\left(-yz\,dx - \frac{z^2}{2}\,dy\right)$$

$$\iint_S \omega = \oint_C -yz\,dx - \frac{z^2}{2}\,dy = 0$$




$$C(t) = \langle \cos t, \sin t, 0 \rangle$$