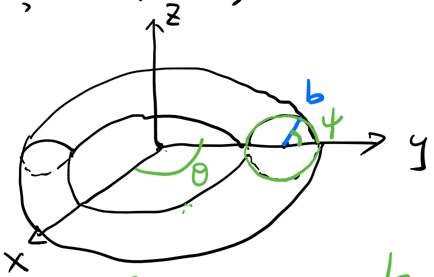
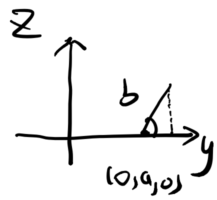
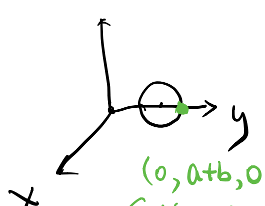
Ex1: Torus $\begin{cases} x = (a+b\cos\psi)\cos\theta \\ y = (a+b\cos\psi)\sin\theta \\ z = b\sin\psi \end{cases}$, $0 \le \psi \le 2\pi$, $0 \le \theta \le 2\pi$, $a>0$, $b>0$, $b<a$





$$\begin{array}{c} \\ T_\psi = \\ T_\theta = \end{array} \left| \begin{array}{ccc} i & j & k \\ \langle (-b\sin\psi)\cos\theta, & (-b\sin\psi)\sin\theta, & b\cos\psi \rangle \\ \langle (a+b\cos\psi)(-\sin\theta), & (a+b\cos\psi)\cos\theta, & 0 \rangle \end{array} \right|$$

$$\begin{aligned} T_\psi \times T_\theta = {} & i\left[-b(a+b\cos\psi)\cos\psi\cos\theta\right] \\ & -j\left[-b(a+b\cos\psi)(-\sin\theta)\cdot\cos\psi\right] \\ & +k\left[-b(a+b\cos\psi)\sin\psi\underline{\cos^2\theta} - b(a+b\cos\psi)\sin\psi\underline{\sin^2\theta}\right] \\ = {} & -b(a+b\cos\psi)\langle \cos\psi\cos\theta, \cos\psi\sin\theta, \sin\psi \rangle \end{aligned}$$

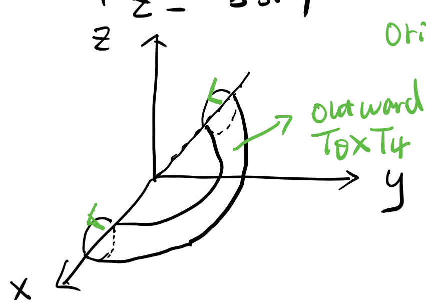


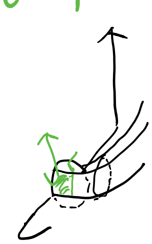
$(0, a+b, 0)$

$\begin{cases} \psi = 0 \\ \theta = \frac{\pi}{2} \end{cases} \Rightarrow T_\psi \times T_\theta = \underbrace{-b(a+b)\langle 0, 1, 0 \rangle}_{\text{inward normal vector}}$

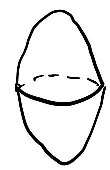
Ex 2: S $\begin{cases} x = (a+b\cos\psi)\cos\theta \\ y = (a+b\cos\psi)\sin\theta \\ z = b\sin\psi \end{cases}$, $0 \le \theta \le \pi$, $0 \le \psi \le 2\pi$

Orientation: $T_\theta \times T_\psi$





Ex 3: Ellipsoid $x^2+y^2+\frac{1}{4}z^2=\underline{\underline{1}}$



Upper half S_1 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 2\sqrt{1-r^2} \end{cases}$ $\quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 1$

Lower half S_2 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = -2\sqrt{1-r^2} \end{cases}$ $\quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 1$

Ex4: Compute Volume of the 3D inside the ellipsoid above.

$$\text{Volume} = \iiint_V 1\, dxdydz = \int_0^{2\pi}\int_0^1 \left[\int_{-2\sqrt{1-r^2}}^{2\sqrt{1-r^2}} 1\, dz\right] r\,dr\,d\theta$$

$$= 2\pi \int_0^1 4\sqrt{1-r^2}\, \underbrace{r\,dr}_{d\left(\frac{r^2}{2}\right)}$$

$$= 4\pi \int_0^1 \sqrt{1-r^2}\, dr^2$$

$$= 4\pi \int_0^1 \sqrt{1-s}\, ds$$

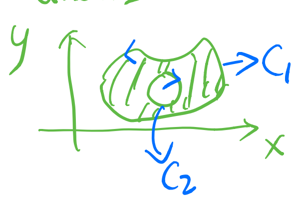
$(1-s=t \Rightarrow s=1-t \Rightarrow ds=-dt)$

$$= 4\pi \int_1^0 \sqrt{t}\,(-dt)$$

$$= 4\pi \int_0^1 \sqrt{t}\, dt = 4\pi\left[\frac{2}{3}t^{\frac{3}{2}}\Big|_0^1\right] = \frac{8\pi}{3}.$$

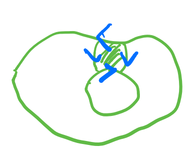
HW Problem P_{42} 5(a)

Green's Theorem $\iint_D [Q_x - P_y]\, dxdy = \oint_C P dx + Q dy$



$C = C_1 \cup C_2$

$\oint_C = \underline{\oint_{C_1} + \oint_{C_2}}$



$P = -\frac{1}{2}y, \quad Q = \frac{1}{2}x$

$Q_x - P_y = 1$

$$\Rightarrow \iint_D 1\, dxdy = \oint_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$$

Gauss/Divergence Theorem

$$\iiint_V \nabla\cdot\langle F,G,H\rangle\,dx\,dy\,dz = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$\langle F,G,H\rangle = \frac{1}{3}\langle x,y,z\rangle$$

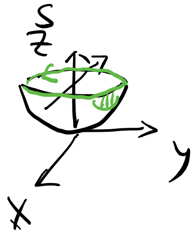
$$\Rightarrow \iiint_V 1\,dx\,dy\,dz = \iint_S \frac{1}{3}\langle x,y,z\rangle\cdot\vec{n}\,dS$$

Ex 5: $\omega = (z\cos x + yz)\,dy\wedge dz + \cos xz\,dz\wedge dx + \frac{z^2}{2}\sin x\,dx\wedge dy$

$S: \begin{cases} z = x^2+y^2 \\ 0\le x^2+y^2\le 1 \end{cases}$ with upward normal.

$$\iint_S \omega$$

Sol:



$S \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases}$ $\quad 0\le\theta\le 2\pi, \quad 0\le r\le 1$

$$\vec{F} = \langle z\cos x + yz,\ \cos xz,\ \frac{z^2}{2}\sin x\rangle$$

$$\nabla\cdot\vec{F} = -z\sin x + z\sin x = 0 \Rightarrow d\omega = 0$$

$\Rightarrow \omega$ is exact

(Poincare Lemma) $\Rightarrow \omega = d\left(-\frac{\sin(xz)}{x}\,dx + \left(\frac{z^2}{2}\cos x + \frac{z^2}{2}y\right)dy\right)$
$= d\alpha$

$$\iint_S \omega = \iint_S d\alpha = \oint_C \alpha$$

$C = \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 1 \end{cases}$ $\quad 0\le\theta\le 2\pi$

$$\iint_{-S+S_1} \omega = \iiint_V d\omega = 0$$

$\rightarrow$

$$\Rightarrow \iint_{-S} \omega + \iint_{S_1} \omega = 0$$

$$\Rightarrow \iint_{S} \omega = \underline{\underline{\iint_{S_1} \omega}} = \iint_{S_1} \vec{F} \cdot \underbrace{\vec{n}}_{\langle 0,0,1 \rangle} dS$$

$$\underbrace{\vec{F} \cdot \vec{n} = \frac{z^2}{2} \sin x}$$