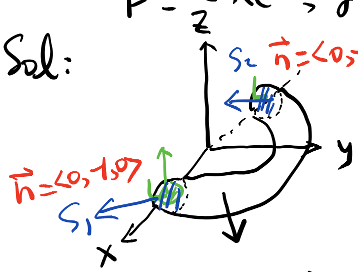
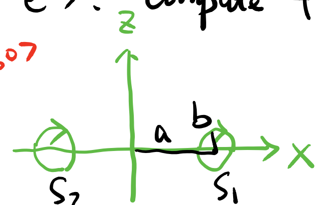
Ex 1: Right half of torus with outward normal

$$S \begin{cases} x = (a+b\cos\psi)\cos\theta \\ y = (a+b\cos\psi)\sin\theta \\ z = b\sin\psi \end{cases} \qquad \begin{array}{l} a>b>0 \\ 0\le\psi\le 2\pi \\ 0\le\theta\le\pi \end{array}$$

$\vec{F} = \langle xe^z, y, -e^z\rangle$. Compute flux of $\vec{F}$ through S.

Sol:





$$S_1 \begin{cases} x = a + r\cos\theta \\ y = 0 \\ z = r\sin\theta \end{cases} \qquad \begin{array}{l} 0\le r\le b \\ 0\le\theta\le 2\pi \end{array}$$

$$S_2 \begin{cases} x = -a + r\cos\theta \\ y = 0 \\ z = r\sin\theta \end{cases} \qquad \begin{array}{l} 0\le r\le b \\ 0\le\theta\le 2\pi \end{array}$$

Flux of $\vec{F}$ is $\iint_S \vec{F}\cdot\vec{n}\,dS$

$$\nabla\cdot\vec{F} = \nabla\cdot\langle xe^z, y, -e^z\rangle = e^z + 1 - e^z = 1$$

Let S_1 and S_2 be two disks with outward normal s.t. $S+S_1+S_2$ is the boundary of solid region defined by right half of torus.

Gauss Theorem $\Rightarrow \iint_{S+S_1+S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dxdydz = \iiint_V 1\,dxdydz$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,dS = -\iint_{S_1} \underbrace{\vec{F}\cdot\vec{n}}_{-y}\,dS - \iint_{S_2} \underbrace{\vec{F}\cdot\vec{n}}_{-y}\,dS + \iiint_V 1\,dxdydz$$

$$= \underbrace{\iint_{S_1} y\,dS}_{0} + \underbrace{\iint_{S_2} y\,dS}_{0} + \iiint_V 1\,dxdydz$$

$$= \iiint_V 1\,dxdydz$$

$$= \iiint_V \nabla\cdot\langle 0,0,z\rangle\,dxdydz$$

$$(\text{Gauss Thm}) = \iint_{S+S_1+S_2} \langle 0,0,z\rangle\cdot\vec{n}\,dS$$

$$= \iint_S \langle 0,0,z\rangle \cdot \vec{n}\, dS + \iint_{S_1} \langle 0,0,z\rangle \cdot \underbrace{\vec{n}}_{\langle 0,-1,0\rangle} dS + \iint_{S_2} \langle 0,0,z\rangle \cdot \vec{n}\, dS$$

$$= \iint_S \langle 0,0,z\rangle \cdot \vec{n}\, dS$$

$$= \iint_D \langle 0,0,z\rangle \cdot \underbrace{T_\theta \times T_\psi}_{\text{(HW\#11 Solutions)}} \, d\theta\, d\psi$$

$$\left(\begin{aligned} T_\theta \times T_\psi &= (a+b\cos\psi) b \langle \cos\psi\cos\theta, \cos\psi\sin\theta, \sin\psi\rangle \\ z &= b\sin\psi \end{aligned} \right)$$

$$= \int_0^{2\pi}\int_0^{\pi} b\sin\psi\,(a+b\cos\psi)\, b\sin\psi\, d\theta\, d\psi$$

$$= b^2\cdot\pi \int_0^{2\pi} \sin^2\psi\,(a+b\cos\psi)\, d\psi$$

$$= b^2\cdot\pi\left[a\int_0^{2\pi}\sin^2\psi\, d\psi + b\int_0^{2\pi}\sin^2\psi\,\underline{\cos\psi\, d\psi}\right]$$

$$= b^2\pi\left[a\int_0^{2\pi}\frac{1-\cos 2\psi}{2}d\psi + b\int_0^{2\pi}\sin^2\psi\, d\sin\psi\right]$$

We can also compute the volume directly:

$$\iiint_V 1\, dx\,dy\,dz = \iiint_V 1\, \underline{dx\wedge dy\wedge dz} = \iiint \underbrace{\boxed{?}}_{\text{Jacobian}} dR\wedge d\psi\wedge d\theta$$

$$V: \begin{cases} x = (a+R\cos\psi)\cos\theta \\ y = (a+R\cos\psi)\sin\theta \\ z = R\sin\psi \end{cases}, \quad \boxed{\begin{aligned} 0 &\le R \le b \\ 0 &\le \psi \le 2\pi \\ 0 &\le \theta \le \pi \end{aligned}}$$

$$dx = \cos\psi\cos\theta\, dR - R\sin\psi\cos\theta\, d\psi - (a+R\cos\psi)\sin\theta\, d\theta$$

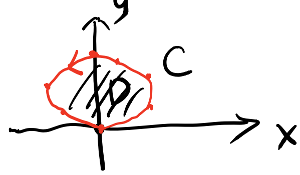
$$\text{Jacobian: } dx\wedge dy\wedge dz = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \psi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \psi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \psi} & \frac{\partial z}{\partial \theta} \end{vmatrix} dR\wedge d\psi\wedge d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad dx\wedge dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\wedge d\theta$$

$$= r\, dr\wedge d\theta$$

Ex2: Area enclosed by $\underline{r = \sin\theta,\ 0 \le \theta \le \pi}$.

Sol:



$$\text{Area}(D) = \iint_D 1\, dx dy = \int_C x\, dy$$

$$(P = 0,\ Q = x)$$

$$\iint_D (Q_x - P_y)\, dx dy = \int_C P dx + Q dy$$

$$\left(\int_M d\omega = \int_{\partial M} \omega \right)$$

$$\text{Area} = \int_C x\, dy = \int_0^{\pi} \frac{1}{2}\sin^2(2\theta)\, d\theta = \int_0^{\pi} \frac{1-\cos(4\theta)}{4} d\theta$$

$$C: \begin{cases} x = r\cos\theta = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \\ y = r\sin\theta = \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \end{cases} \qquad 0 \le \theta \le \pi$$

$$dy = y'(\theta)\, d\theta = \sin(2\theta)\, d\theta$$