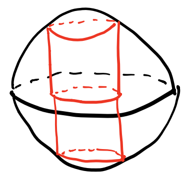
1. $x^2+y^2+z^2 \le 4$

$x^2+y^2 \le 1$



$$V: \begin{cases} x^2+y^2 \le 1 \\ -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2} \end{cases} \qquad V \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2} \end{cases}$$

① $$\text{Volume}(V) = \iiint_V 1 \, \underline{dx\,dy}\,dz = \iiint_V 1 \, \underline{r\,dr\,d\theta}\,dz$$

$$= \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 \, dz \, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 2\sqrt{4-r^2}\, \underbrace{r\,dr}_{d\left(\frac{r^2}{2}\right)}$$

$$= 2\pi \int_0^1 2\sqrt{4-t}\, d\frac{t}{2}$$

$$= 2\pi \int_0^1 \sqrt{\underbrace{4-t}_{u}}\, dt$$

$(u = 4-t, \quad dt = -du)$

$$= 2\pi \int_4^3 \sqrt{u}\,(-du)$$

$$= 2\pi \int_3^4 \sqrt{u}\,du = 2\pi\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_3^4$$

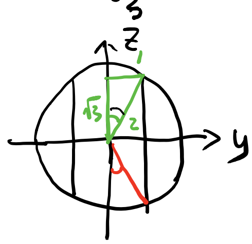
② S_1

S_2

$$S_1 \begin{cases} x = 2\sin\phi\cos\theta \\ y = 2\sin\phi\sin\theta \\ z = 2\cos\phi \end{cases} \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{6} \end{matrix}$$

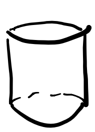
$$S_3 \begin{cases} 0 \le \theta \le 2\pi \\ \frac{5\pi}{6} \le \theta \le \pi \end{cases}$$

S_3



$$\text{Area}(S_1) = \iint_{S_1} dS = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \|T_\phi \times T_\theta\|\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} 4\sin\phi\,d\phi\,d\theta$$

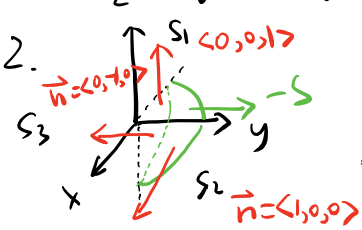


$$S_2 \begin{cases} x^2+y^2 = 1 \\ -\sqrt{3} \le z \le \sqrt{3} \end{cases} \qquad S_2 \begin{cases} x = \cos u \\ y = \sin u \\ z = v \end{cases} \qquad \begin{matrix} 0 \le u \le 2\pi \\ -\sqrt{3} \le z \le \sqrt{3} \end{matrix}$$

$$2\sqrt{3}\cdot 2\pi$$

$$\iint_{S_2} dS = \int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} \|T_u \times T_v\|\, du\, dv = \int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} 1\, du\, dv = 2\sqrt{3}\cdot 2\pi$$

$$\begin{array}{llll} & i & j & k \\ T_u = & \langle -\sin u, & \cos u, & 0\rangle \\ T_v = & \langle\ 0, & 0, & 1\rangle \\ T_u\times T_v = & \langle \cos u, & \sin u, & 0\rangle \end{array}$$

2. 

$$\iint_S dx\wedge dy = \iint_S \underbrace{\langle 0,0,1\rangle}_{\vec{F}}\cdot\vec{n}\, dS$$

$$\nabla\cdot\langle 0,0,1\rangle = 0$$

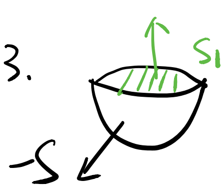
Gauss Theorem $\Rightarrow \displaystyle\iint_{S_1+S_2+S_3-S} \langle 0,0,1\rangle\cdot\vec{n}\, dS = \iiint_V \nabla\cdot\langle 0,0,1\rangle\, dx\,dy\,dz$

$$= 0$$

$$\Rightarrow \iint_S \langle 0,0,1\rangle\cdot\vec{n}\, dS = \iint_{S_1+S_2+S_3} \underbrace{\langle 0,0,1\rangle\cdot\vec{n}}\, dS$$

$$= \iint_{S_1} \langle 0,0,1\rangle\cdot\vec{n}\, dS$$

$$= \iint_{S_1} 1\, dS = \text{Area}(S_1)$$

3. 

① $\displaystyle\iint_S xy\, dS$

② $\displaystyle\iint_S \underbrace{\langle \cos z^3,\ e^{x^2z^2},\ z\rangle}_{\vec{F}}\cdot\vec{n}\, dS$

$\nabla\cdot\vec{F} = 1$

$$S\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases} \qquad 0\le r\le 1 \qquad 0\le\theta\le 2\pi$$

$$\begin{array}{llll} & i & j & k \\ T_r = & \langle \cos\theta, & \sin\theta, & 2r\rangle \\ T_\theta = & \langle -r\sin\theta, & r\cos\theta, & 0\rangle \end{array}$$

$$T_r\times T_\theta = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle$$
$$= r\langle -2r\cos\theta, -r\sin\theta, 1\rangle$$

$$\|T_r\times T_\theta\| = r\sqrt{4r^2+1}$$

① $\iint_S xy\,dS = \int_0^{2\pi}\int_0^1 r^2\cos\theta\sin\theta\, r\sqrt{4r^2+1}\,dr\,d\theta$

$= \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^1 r^2\sqrt{4r^2+1}\,\underline{\underline{r\,dr}}\right)$

$= \int_0^{2\pi}\left(\frac{1}{2}\sin 2\theta\right)d\theta \; \underbrace{\int_0^1 r^2\sqrt{4r^2+1}\left(d\frac{r^2}{2}\right)}$

$\frac{1}{2}\int_0^1 t\sqrt{4t+1}\,dt$

$\left(u=4t+1,\; dt=\frac{1}{4}du,\; t=\frac{u-1}{4}\right)$

$= \frac{1}{2}\int_1^5 \frac{u-1}{4}\sqrt{u}\left(\frac{1}{4}du\right)$

$= \frac{1}{32}\int_1^5 \left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$

$\vec{n}=\langle 0,0,1\rangle$ $= \frac{1}{32}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]\Big|_1^5$

② 

Gauss Thm $\Rightarrow \iint_{S_1-S}\vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dxdydz$

$= \iiint_V 1\,dxdydz$

$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS - \iiint_V 1\,dxdydz$

$= \iint_{S_1} z\,dS - \iiint_V 1\,dxdydz$

$S_1 \begin{cases} x = r\cos\theta & 0\le\theta\le 2\pi \\ y = r\sin\theta & 0\le r\le 1 \\ z = 1 & \end{cases}$

$= \iint_{S_1} 1\,dS - \iiint_V 1\,dxdydz$

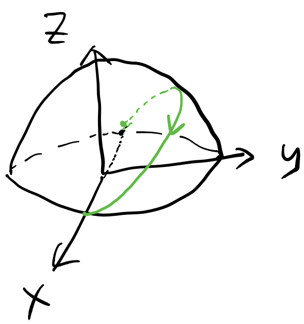
$= \pi^2 - \iiint_V 1\,dxdydz$

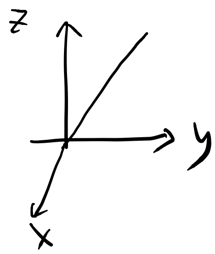
$$V: \begin{cases} x^2+y^2 \le z \le 1 \\ x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{aligned}$$

$$\iiint_V 1\, \underline{dx\,dy\,dz} = \iiint_V 1\, \underline{r\,dr\,d\theta\,dz} = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^1 r(1-r^2)\,dr$$

4. $y = z = \sqrt{1-x^2-y^2}$ $\begin{cases} ① & z = \sqrt{1-x^2-y^2} \\ ② & y = z \end{cases}$





1) $C \begin{cases} x = t \\ y = \sqrt{\frac{1-t^2}{2}} \\ z = \sqrt{\frac{1-t^2}{2}} \end{cases}$

$y = \sqrt{1-t^2-y^2} \Rightarrow y^2 = 1-t^2-y^2$

$\Rightarrow 2y^2 = 1-t^2$

$\Rightarrow y = \sqrt{\frac{1-t^2}{2}}$

2) $C \begin{cases} x = \sin\theta \\ y = \frac{1}{\sqrt{2}}\cos\theta \\ z = \frac{1}{\sqrt{2}}\cos\theta \end{cases}$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\text{Length of } C = \int_C ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2 + [z'(\theta)]^2}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta + \frac{1}{2}\sin^2\theta + \frac{1}{2}\sin^2\theta}\, d\theta$$

$$= \pi.$$