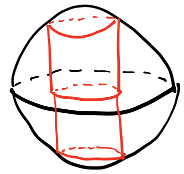
1. $x^2+y^2+z^2 \le 4$

$x^2+y^2 \le 1$



$$V: \begin{cases} x^2+y^2 \le 1 \\ -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2} \end{cases} \qquad V \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2} \end{cases}$$

① $$\text{Volume}(V) = \iiint_V 1 \, \underline{dx\,dy\,dz} = \iiint_V 1 \, \underline{r\,dr\,d\theta}\, dz$$

$$= \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 \, dz \, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 2\sqrt{4-r^2}\, \underbrace{r\,dr}_{d(\frac{r^2}{2})}$$

$$= 2\pi \int_0^1 2\sqrt{4-t}\, d\frac{t}{2}$$

$$= 2\pi \int_0^1 \sqrt{\underbrace{4-t}_{u}}\, dt$$

$(u = 4-t, \quad dt = -du)$

$$= 2\pi \int_4^3 \sqrt{u}\,(-du)$$

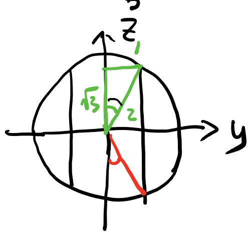
$$= 2\pi \int_3^4 \sqrt{u}\,du = 2\pi\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_3^4$$

②

$$S_1 \begin{cases} x = 2\sin\phi\cos\theta \\ y = 2\sin\phi\sin\theta \\ z = 2\cos\phi \end{cases} \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{6} \end{matrix}$$

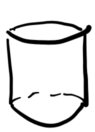
$$S_3 \begin{cases} 0 \le \theta \le 2\pi \\ \frac{5\pi}{6} \le \theta \le \pi \end{cases}$$

S_3



$$\text{Area}(S_1) = \iint_{S_1} dS = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \|T_\phi \times T_\theta\|\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} 4\sin\phi\, d\phi\, d\theta$$

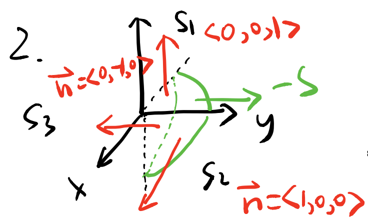


$$S_2 \begin{cases} x^2+y^2 = 1 \\ -\sqrt{3} \le z \le \sqrt{3} \end{cases} \qquad S_2 \begin{cases} x = \cos u \\ y = \sin u \\ z = v \end{cases} \qquad \begin{matrix} 0 \le u \le 2\pi \\ -\sqrt{3} \le z \le \sqrt{3} \end{matrix}$$

$$\iint_{S_2} dS = \int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} \|T_u \times T_v\|\, du\, dv = \int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} 1\, du\, dv = \overset{2\sqrt{3}\cdot 2\pi}{2\sqrt{3}\cdot 2\pi}$$

$$\begin{array}{lccc} & i & j & k \\ T_u = & \langle -\sin u, & \cos u, & 0\rangle \\ T_v = & \langle 0, & 0, & 1\rangle \\ T_u\times T_v = & \langle \cos u, & \sin u, & 0\rangle \end{array}$$

2.



$$\iint_S dx\wedge dy = \iint_S \underbrace{\langle 0,0,1\rangle}_{\vec{F}}\cdot\vec{n}\, dS$$

$$\nabla\cdot\langle 0,0,1\rangle = 0$$

Let V be the 3D region enclosed by S_1, S_2, S_3 and S.

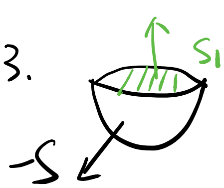
$$\text{Gauss Theorem} \Rightarrow \iint_{S_1+S_2+S_3-S} \langle 0,0,1\rangle\cdot\vec{n}\, dS = \iiint_V \nabla\cdot\langle 0,0,1\rangle\, dx\,dy\,dz = 0$$

$$\Rightarrow \iint_S \langle 0,0,1\rangle\cdot\vec{n}\, dS = \iint_{S_1+S_2+S_3} \langle 0,0,1\rangle\cdot\vec{n}\, dS$$

$$= \iint_{S_1} \langle 0,0,1\rangle\cdot\vec{n}\, dS$$

$$= \iint_{S_1} 1\, dS = \text{Area}(S_1)$$

3.



① $\iint_S xy\, dS$

② $\iint_S \underbrace{\langle \cos z^3, e^{x^2z^2}, z\rangle}_{\vec{F}}\cdot\vec{n}\, dS$ $\quad \nabla\cdot\vec{F} = 1$

$$S\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases} \qquad 0\le r\le 1, \quad 0\le\theta\le 2\pi$$

$$\begin{array}{lccc} & i & j & k \\ T_r = & \langle \cos\theta, & \sin\theta, & 2r\rangle \\ T_\theta = & \langle -r\sin\theta, & r\cos\theta, & 0\rangle \end{array}$$

$$T_r\times T_\theta = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle$$
$$= r\langle -2r\cos\theta, -r\sin\theta, 1\rangle$$

$$\|T_r\times T_\theta\| = r\sqrt{4r^2+1}$$

① $\iint_S xy\,dS = \int_0^{2\pi}\int_0^1 r^2\cos\theta\sin\theta\, r\sqrt{4r^2+1}\,dr\,d\theta$

$= \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^1 r^2\sqrt{4r^2+1}\,\underline{r\,dr}\right)$

$= \int_0^{2\pi}\left(\frac{1}{2}\sin 2\theta\right)d\theta \; \underbrace{\int_0^1 r^2\sqrt{4r^2+1}\,\left(d\frac{r^2}{2}\right)}$

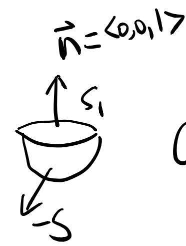
$\frac{1}{2}\int_0^1 t\sqrt{4t+1}\,dt$

$\left(u=4t+1,\; dt=\frac{1}{4}du,\; t=\frac{u-1}{4}\right)$

$= \frac{1}{2}\int_1^5 \frac{u-1}{4}\sqrt{u}\left(\frac{1}{4}du\right)$

$= \frac{1}{32}\int_1^5 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$

$= \frac{1}{32}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]\Big|_1^5$

② 

Gauss Thm $\Rightarrow \iint_{S_1 - S}\vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dx\,dy\,dz$

$= \iiint_V 1\,dx\,dy\,dz$

$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS - \iiint_V 1\,dx\,dy\,dz$

$= \iint_{S_1} z\,dS - \iiint_V 1\,dx\,dy\,dz$

$S_1 \begin{cases} x = r\cos\theta & 0\le\theta\le 2\pi \\ y = r\sin\theta & 0\le r\le 1 \\ z = 1 \end{cases}$

$= \iint_{S_1} 1\,dS - \iiint_V 1\,dx\,dy\,dz$

$= \pi^2 - \iiint_V 1\,dx\,dy\,dz$

$$V:\begin{cases} x^2+y^2 \le z \le 1 \\ x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{array}{l} 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{array}$$

$$\iiint_V 1\,\underline{dx\,dy\,dz} = \iiint_V 1\,\underline{r\,dr\,d\theta\,dz} = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dz\,dr\,d\theta$$

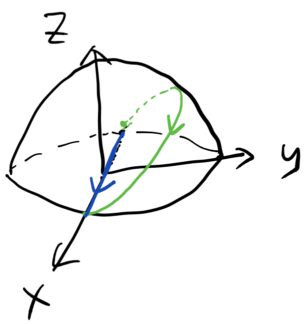
$$= 2\pi\int_0^1 r(1-r^2)\,dr$$

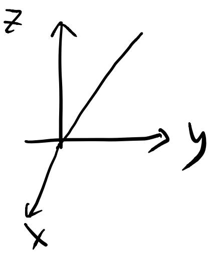
4. $y = z = \sqrt{1-x^2-y^2}$ $\begin{cases} ① \ z=\sqrt{1-x^2-y^2} \\ ② \ y = z \end{cases}$

$$\int_C \langle e^{y^2-z^2}, 2xye^{y^2-z^2}, -2xze^{y^2-z^2}\rangle\cdot d\vec{s}$$

$$\int_C \vec{F}\cdot d\vec{s}$$

$$\nabla\times\vec{F} = \vec{0}$$





1) $C\begin{cases} x = t \\ y = \sqrt{\frac{1-t^2}{2}} \\ z = \sqrt{\frac{1-t^2}{2}} \end{cases}$

$y = \sqrt{1-t^2-y^2} \Rightarrow y^2 = 1-t^2-y^2$

$\Rightarrow 2y^2 = 1-t^2$

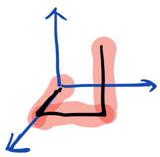
$\Rightarrow y = \sqrt{\frac{1-t^2}{2}}$

2) $C\begin{cases} x = \sin\theta \\ y = \frac{1}{\sqrt{2}}\cos\theta \\ z = \frac{1}{\sqrt{2}}\cos\theta \end{cases}$ $\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

Length of $C = \int_C ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{[x'(\theta)]^2+[y'(\theta)]^2+[z'(\theta)]^2}\,d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{2}\sin^2\theta}\,d\theta$$

$$= \pi.$$



$$f(x,y,z) = \int_0^x F(t,0,0)\,dt + \int_0^y G(x,t,0)\,dt + \int_0^z H(x,y,t)\,dt$$

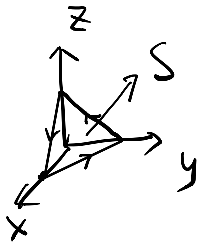
$$= \int_0^x 1\,dt + \int_0^y 2xt\,e^{t^2}\,dt + \int_0^z (-2xt)\,e^{y^2-t^2}\,dt$$

$$= x + x\int_0^y e^{t^2}\underbrace{2t\,dt}_{d(t^2)} - xe^{y^2}\int_0^z e^{-t^2}\underbrace{2t\,dt}_{d(t^2)}$$

$$= x + x\int_0^{y^2} e^{v}\,dt - x\,e^{y^2}\int_0^{z^2} e^{-v}\,dv$$

$$= x\,e^{y^2-z^2}$$

5.



$$\int_C \alpha = \iint_S d\alpha$$

$$\vec{F} = \langle z^2, y^2, x\rangle + \nabla(-\cos(xyz))$$

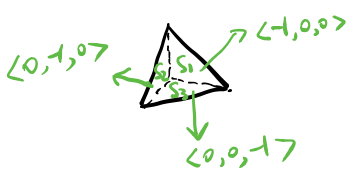
$$\nabla\times\vec{F} = \nabla\times\langle z^2, y^2, x\rangle + \underbrace{\nabla\times(\nabla(-\cos(xyz)))}_{d^2(-\cos(xyz))}$$

$$= \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ z^2 & y^2 & x \end{vmatrix}$$

$$= i\cdot 0 - j(1-2z) + k\cdot 0$$

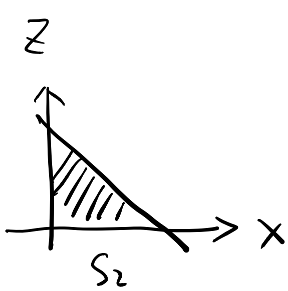
$$= \langle 0, 2z-1, 0\rangle$$

$$\iint_S d\alpha = \iint_S \langle 0, 2z-1, 0\rangle\cdot\vec{n}\,dS$$



$$\iint_{S+S_1+S_2+S_3} d\alpha = \iiint_V d(d\alpha) = 0$$

$$\Rightarrow \iint_S = -\iint_{S_1+S_2+S_3} d\alpha = -\iint_{S_1+S_2+S_3} \langle 0, 2z-1, 0\rangle\cdot\vec{n}\,dS$$



$$= -\iint_{S_2} \langle 0, 2z-1, 0\rangle\cdot\langle 0,-1,0\rangle\,dS$$

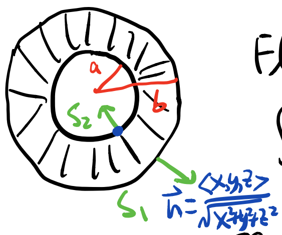
$$= \iint_{S_2} (2z-1)\,dS$$

$$S_2 \begin{cases} x = u \\ y = 0 \\ z = v \end{cases} \qquad \begin{matrix} u+v \le 1 \\ u \ge 0 \\ v \ge 0 \end{matrix} \qquad \|T_u \times T_v\| = 1$$

$$\iint_{S_2} (2z-1)\,dS = \int_0^1 \left[\int_0^{1-v} (2v-1)\,du\right] dv$$

$$= \int_0^1 (1-v)(2v-1)\,dv$$

6. ①



Flux of $\vec{F}$ through the boundary is

$$\iint_{S_1+S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} + \iint_{S_2}$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S_2} \frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{-\langle x,y,z\rangle}{(x^2+y^2+z^2)^{\frac{1}{2}}}\,dS$$

$$\vec{n} = \frac{-\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}$$

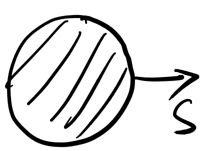
$$= \iint_{S_2} -\frac{1}{x^2+y^2+z^2}\,dS = \iint_{S_2} -\frac{1}{a^2}\,dS = -\frac{1}{a^2}\iint \|T_\phi \times T_\theta\|\,d\phi\,d\theta$$

$$= -4\pi$$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = 4\pi$$ Can also use Gauss Thm.

---

②



$$\iint_S \vec{F}\cdot\vec{n}\,dS = 4\pi$$

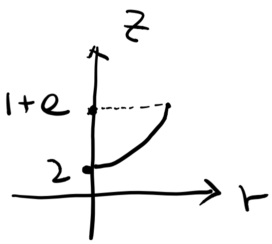
Cannot use Gauss Theorem

8. (a) $\nabla\cdot\langle -x, y-z, z\rangle = 0$ ✓ $\omega = d\alpha$

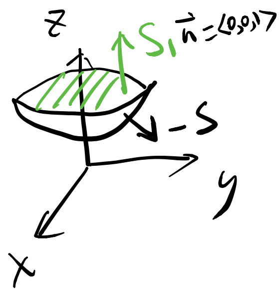
(b) $\nabla\cdot\langle x, x-z, z\rangle = 2$ ✗

(c) $\nabla\cdot\langle 2yz, 3x^2z, x\rangle = 0$ ✓

9. $$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \cos^2 r^2 + e^{r^2} \end{cases} \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{matrix}$$



i j k



$$\iint_S \langle 0,0,\sin^2(x^2+y^2)+z\rangle\cdot\vec{n}\,dS$$

Gauss Thm $\Rightarrow$

$$\iint_{S_1-S}\vec{F}\cdot\vec{n}\,dS=\iiint_V \nabla\cdot\vec{F}\,dxdydz$$

$$=\iiint_V 1\,dxdydz$$

$T_r=\langle\cos\theta,\sin\theta,2\cos r^2(-\sin r^2)2r+e^{r^2}\cdot 2r\rangle$

$T_\theta=\langle -r\sin\theta, r\cos\theta, 0\rangle$

$T_r\times T_\theta=\vec{i}\cdot * + \vec{j}\cdot * + \vec{k}\,r$

$$\iint_S \vec{F}\cdot\vec{n}\,dS=\iint_D \vec{F}\cdot T_r\times T_\theta\,drd\theta$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}=\iint_{S_1}\vec{F}\cdot\vec{n}\,dS-\iiint_V 1\,dxdydz=\iint_D(\sin^2(r^2)+z)\cdot r\,drd\theta$$

$$=\iint_D\left[\sin^2(r^2)+(\cos^2 r^2+e^{r^2})\right]r\,drd\theta$$

$$=\iint_{S_1} z\,dS-\iiint_V 1\,dxdydz$$

$$\iint_{S_1} z\,dS=\iint_{S_1}(\cos^2 1+e)\,dS=(\cos^2 1+e)\iint_{S_1}dS=(\cos^2 1+e)\pi$$

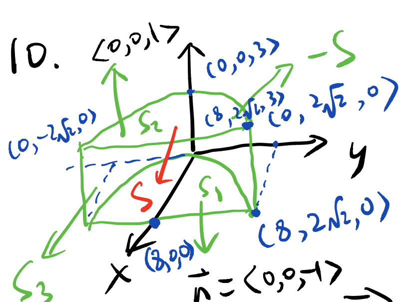
$$\iiint_V 1\,dxdydz=\iiint_V 1\,r\,drd\theta dz=\int_0^{2\pi}\int_0^1\left(\int_{\cos^2 r^2+e^{r^2}}^{\cos^2 1+e} r\,dz\right)dr\,d\theta$$

$V:\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ \cos^2 r^2+e^{r^2}\le z\le \cos^2 1+e\end{cases}$ $\quad 0\le\theta\le 2\pi$, $0\le r\le 1$

$$=2\pi\int_0^1(\cos^2 1+e-\cos^2 r^2-e^{r^2})\underbrace{r\,dr}_{d\left(\frac{r^2}{2}\right)}$$

$$=\pi\int_0^1(\cos^2 1+e-\cos^2 t-e^t)\,dt$$

10.



$\vec{n}=\langle 1,0,0\rangle$

$S_3\begin{cases} x=8 \\ y=u \\ z=v\end{cases}$ $\quad -2\sqrt{2}\le u\le 2\sqrt{2}$, $0\le v\le 3$

$$\iint_S\langle y^2z,(x+1)^z,0\rangle\cdot\vec{n}\,dS$$

$\nabla\cdot\langle y^2z,(x+1)^z,0\rangle=0\Rightarrow$ 2-form is exact

$\Rightarrow$ 1) Stokes Theorem $\iint_S d\alpha=\int_C\alpha$

Use Poincare's Lemma to find α.

2) Gauss Theorem

$$\iint_{-S+S_1+S_2+S_3} \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dx\,dy\,dz = 0$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1+S_2+S_3} \vec{F}\cdot\vec{n}\,dS$$

$$= \iint_{S_3} \vec{F}\cdot\vec{n}\,dS$$

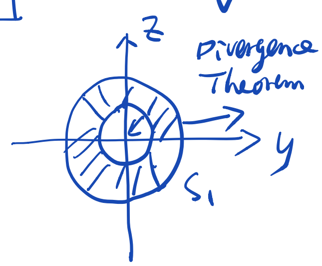
$$= \iint_{S_3} y^2 z\,dS$$

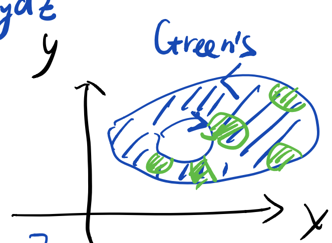
---

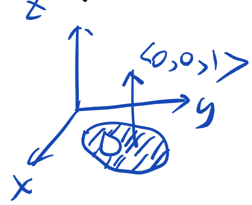
① $$\iiint_V \underbrace{\boxed{}}_{\nabla\cdot\vec{F}}\,dx\,dy\,dz = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$\iiint_V 1\,dx\,dy\,dz = \iiint_V \nabla\cdot\langle x,0,0\rangle\,dx\,dy\,dz = \iint_S \langle x,0,0\rangle\,dx\,dy\,dz$$

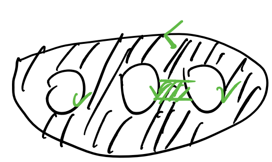
② $$\iint_{\boxed{S}} \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dx\,dy\,dz$$





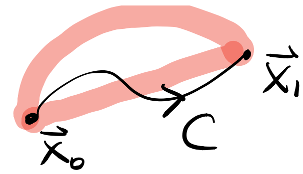


$$\iint_S \omega = \iint_S d\alpha = \oint_C \alpha$$



$\alpha = F\,dx + G\,dy + H\,dz$

$d\alpha = 0 \iff \nabla \times \langle F, G, H \rangle = \vec{0}$



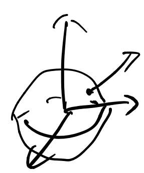
α is exact $\Rightarrow$ ① $\int_C \alpha = \int_C df = f(\vec{x}_1) - f(\vec{x}_0)$

② $\int_C \alpha$ is path-independent

$$\iint_S F\,dy\wedge dz + G\,dz\wedge dx + H\,dx\wedge dy = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_V \nabla\cdot\vec{F}\,dx\,dy\,dz$$

$$\vec{n} = \frac{T_u \times T_v}{\|T_u \times T_v\|}$$



$\vec{n} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$