MA 362
Spring 2020

Homework 8

Due on Mar 25 before 10:30am on blackboard.

1. (30 pts) Find α so that $d\alpha = zdx \wedge dz + dy \wedge dz$. Follow the proof of Poincaré's Lemma to derive α step by step. Do not apply the formulas directly.

2. (30 pts) For the following vector fields $\langle f, g, h\rangle$, dertermine if there exists another vector field $\langle F, G, H\rangle$ so that $\nabla \times \langle F, G, H\rangle = \langle f, g, h\rangle$. If yes, apply the formulas direclty to find what $\langle F, G, H\rangle$ are. If no, explain why.
   (a) $\langle f, g, h\rangle = \langle xe^z + x, -y, -e^z\rangle$.
   (b) $\langle f, g, h\rangle = \langle \cos(xyz), \sin(xyz), \sin(xyz)\rangle$.
   (c) $\langle f, g, h\rangle = \langle \cos(yz), \cos(xz), \cos(xy)\rangle$.

3. (40 pts) Consider a four-variable (x, y, z, w denote the four variables) 3-form:
$$\omega = (yw + xy)dx \wedge dy \wedge dz + yzdx \wedge dy \wedge dw + zdx \wedge dw \wedge dz.$$
   (a) Compute $d\omega$ and show that $d\omega = 0$. (Hint: for wedge product of four abstract vectors, switching any two vectors in any two positions results in a negative sign; if any two are the same, the wedge product is zero.)
   (b) By Poincaré's Lemma, there exists a 2-form α such that $d\alpha = \omega$. Find what α is following these steps:

   Step I: rewrite ω as $\omega = \omega_1 + \omega_2$ with ω_1 containing no dw, and dw appearing in the last position in the wedge products in ω_2, i.e.,
$$\omega_1 = (yw + xy)dx \wedge dy \wedge dz,$$
$$\omega_2 = yzdx \wedge dy \wedge dw - zdx \wedge dz \wedge dw,$$
   then construct a 2-form β by integrating the coefficients in ω_2:
$$\beta = \left(\int_0^w yzdt\right) dx \wedge dy + \left(\int_0^w -zdt\right) dx \wedge dz.$$

   Step II: Finish these integrals and write down what β is. Compute $d\beta$ and find what $d\beta - \omega$ is (yes, it is $d\beta - \omega$ instead of $d\beta + \omega$, but why?).

Step III: Now $d\beta - \omega$ should be something like

$$F(x, y, z)dx \wedge dy \wedge dz,$$

then construct a 2-form γ by

$$\gamma = \left(\int_0^z F(x, y, t)dt \right) dx \wedge dy.$$

Compute $d\gamma$ and show that $d\beta - \omega - d\gamma = 0$.

Last Step: therefore we have $d\beta - d\gamma = \omega$, so $\alpha = \beta - \gamma$. Find what α is. Compute $d\alpha$ to verify that $d\alpha$ is indeed ω.