Example: find Jordan Decomposition for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution:

1. Find characteristic polynomials. $|A - \lambda I| = -(\lambda - 1)^5$ (because $A - \lambda I$ is upper triangular). So we know there is only one eigen-value with algebraic multiplicity 5.

2. Find the eigen-space $N(A - \lambda I)$ for each λ. RREF denotes reduced row echelon form.

$$A - I = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; RREF(A - I) = \begin{bmatrix} 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So

$$N(A - I) = Span \left\{ \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus the geometrical multiplicity is 3. And we have three Jordan blocks for this eigen-value. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ denote these three eigen-vectors.

3. The sizes of the Jordan blocks are related to generalized eigen-vectors. We solve the following three linear systems seperately:

$$(A - I)\mathbf{x} = \mathbf{v}_1,$$

$$(A - I)\mathbf{x} = \mathbf{v}_2,$$

$$(A - I)\mathbf{x} = \mathbf{v}_3.$$

If we can find any solution $\mathbf{x}$, then it's a generalized eigen-vector.

In general, solve the three linear systems. Some of them may not have any solutions. For this example, the matrices are simple thus we can easily see $(A - I)\mathbf{x} = \mathbf{v}_1$ and $(A - I)\mathbf{x} = \mathbf{v}_2$ have no solutions because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not in the column space of $A - I$.

Since $(A - I)\mathbf{x} = \mathbf{v}_1$ and $(A - I)\mathbf{x} = \mathbf{v}_2$ have no solutions, there are no generalized eigen-vectors related to them thus there are two 1×1 Jordan blocks corresponding to eigen-vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Since we know there are three Jordan blocks, so the third Jordan block must be 3×3 for this 5×5 matrix (this is a special case, in general at this step we may not know exactly what sizes they are and we have to find all generalized eigen-vectors first).

Therefore the Jordan form is (unique up to permutation of Jordan blocks):

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Find generalized eigen-vectors. Since we know there is a 3×3 Jordan block, we need to find two generalized eigen-vectors $\mathbf{v}_4$ and $\mathbf{v}_5$ satisfying:

$$(A - I)\mathbf{v}_4 = \mathbf{v}_3$$

$$(A - I)\mathbf{v}_5 = \mathbf{v}_4$$

To find the solutions for $(A-I)\mathbf{x} = \mathbf{v}_3$, find RREF of the augmented matrix $[A-I|\mathbf{v}_3]$:

$$RREF[A - I|\mathbf{v}_1] = \left[\begin{array}{ccccc|c} 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution set for $(A - I)\mathbf{x} = \mathbf{v}_3$ is

$$\left\{ r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$$

We need a vector $\mathbf{v}_4 \in N[(A - I)^2] \cap C(A - I)$. Any vector in the solution set above is in $N[(A - I)^2]$ (if $(A - I)\mathbf{x} = \mathbf{v}_3$ then $(A - I)^2\mathbf{x} = (A - I)\mathbf{v}_3 = \mathbf{0}$).

We pick the solution with $r = t = 0$ and $s = 1$ (the one with $r = s = t = 0$ does not work because that solution is not in C(A-I)).

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next to solve $(A - I)\mathbf{x} = \mathbf{v}_4$ ($\mathbf{v}_4 \in C(A - I)$ ensures we have solutions), we get

$$RREF[A - I|\mathbf{v}_4] = \left[\begin{array}{ccccc|c} 0 & 1 & 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution set for $(A - I)\mathbf{x} = \mathbf{v}_4$ is

$$\left\{ r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$$

Any solution will do, so we pick $r = s = t = 0$. Thus

$$\mathbf{v}_5 = \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

5. Let $P = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4 \ \ \mathbf{v}_5]$, then $A = PJP^{-1}$.
   Let $P_2 = [\mathbf{v}_2 \ \ \mathbf{v}_1 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4 \ \ \mathbf{v}_5]$ then $A = P_2 J P_2^{-1}$.
   Let $P_3 = [\mathbf{v}_3 \ \ \mathbf{v}_4 \ \ \mathbf{v}_5 \ \ \mathbf{v}_1 \ \ \mathbf{v}_2]$, then

$$A = P_3 \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P_3^{-1}.$$

   Let $P_4 = [\mathbf{v}_1 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4 \ \ \mathbf{v}_5 \ \ \mathbf{v}_2]$ or $P_4 = [\mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4 \ \ \mathbf{v}_5 \ \ \mathbf{v}_1]$, then

$$A = P_4 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P_4^{-1}.$$

   These are pretty much all possible Jordan decompositions structures (of course eigen-vectors and generalized ones are not unique, we can always use other eigen-vectors to obtain different P). The point is that the order of $\mathbf{v}_3 \ \ \mathbf{v}_4 \ \ \mathbf{v}_5$ cannot be permutated because $\mathbf{v}_4 \ \ \mathbf{v}_5$ are generalized eigen-vectors.

Remark 1: If we have multiple different eigen-values, apply this method to each eigen-value.

Remark 2: Why do the solution sets of $(A - I)\mathbf{x} = \mathbf{v}_i$ look similar? Recall that solutions to $Ax = b$ (if exist) are solutions to $Ax = 0$ plus a particular solution to $Ax = b$.