Def Indicator function of a set S

$$i_S(x) = \begin{cases} 0, & x \in S \\ +\infty, & x \notin S \end{cases}$$

S is convex $\Rightarrow i_S$ is convex

Example:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} & \|x\|_1 \\ \text{s.t.} & Ax = b \end{cases} \quad (*)$$

$$m\overset{n}{\boxed{\quad A \quad}}\,\boxed{x} = \boxed{b}$$

$S = \{x \in \mathbb{R}^n : Ax = b\}$

or $S = \{x : Ax = b\}$

$(*)$ is equivalent to

$$\min_{x \in \mathbb{R}^n} \|x\|_1 + i_{\{x : Ax = b\}} \quad (**)$$

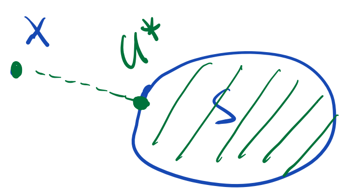
$\boxed{\text{Theorem}}$ The proximal operator of $i_S(x)$ is the projection to S

Proof: $f(x) = i_S(x)$

$$\operatorname{Prox}_f^{\gamma}(x) = \operatorname*{argmin}_u \left[f(u) + \frac{1}{2\gamma}\|u - x\|^2 \right]$$

$$= \operatorname*{argmin}_u \left[i_S(u) + \frac{1}{2\gamma}\|u - x\|^2 \right]$$

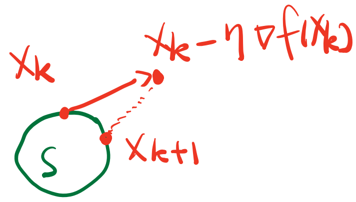
The minimizer is projection of x onto S



$P_S(x)$

Constrained Minization

$$\begin{cases} \min\limits_{x} f(x) \\ \text{s.t. } x \in S \end{cases}$$



$$\Longleftrightarrow \min_{x} [f(x) + g(x)] \qquad g(x) = i_S(x)$$

Projected Gradient Method

$$x_{k+1} = P_S(x_k - \eta \nabla f(x_k))$$
$$= (I + \eta \partial g)^{-1}(x_k - \eta \nabla f(x_k))$$

Forward-Backward Splitting :

for solving $0 \in A(x) + B(x)$

where A & B are operators

$$x_{k+1} = x_k - \eta [A(x_k) + B(x_{k+1})]$$

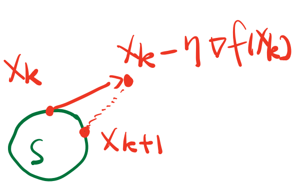
$$(I + \eta B)(x_{k+1}) = x_k - \eta A(x_k)$$

$$x_{k+1} = (I + \eta B)^{-1}(I - \eta A)(x_k)$$

Example : ① Projected Gradient Method

$$\begin{cases} \min\limits_{x} f(x) \\ \text{s.t. } x \in S \end{cases} \Longleftrightarrow \min_{x} f(x) + g(x) \qquad g(x) = i_S(x)$$

$$0 \in \nabla f(x) + \partial g(x)$$



$$A(x) = \nabla f(x), \quad B(x) = \partial g(x)$$

$$x_{k+1} = (I + \eta B)^{-1}(I - \eta A)(x_k)$$

$$= P_S(x_k - \eta \nabla f(x_k))$$

② Proximal Gradient for Composite Optimization

$$\min_x [f(x) + g(x)]$$

$f(x)$ is smooth
$g(x)$ is nonsmooth

$$A(x) = \nabla f(x), \quad B(x) = \partial g(x)$$

$$x_{k+1} = (I + \eta B)^{-1}(I - \eta A)(x_k)$$

$$= (I + \eta \partial g)^{-1}(x_k - \eta \nabla f(x_k))$$

③ Implicit - Explicit (IMEX) for ODE

$$\frac{d}{dt} u(t) = A(u) + B(u)$$

$$\frac{u^{n+1} - u^n}{\Delta t} = \underbrace{A(u^n)}_{\text{explicit}} + \underbrace{B(u^{n+1})}_{\text{implicit}}$$

Example: $u_t = u_x + u_{xx}$

④ LASSO: $\min_x \left(\frac{\lambda}{2} \|Ax - b\|^2 + \|x\|_1 \right)$

ISTA (Iterative Shrinkage-Thresholding Alg)

$$x_{k+1} = T_\gamma [x_k - \gamma \lambda A^T (Ax_k - b)]$$

$$T_\gamma(x)_i = \begin{cases} x_i - \gamma & , \text{ if } x_i > \gamma \\ x_i + \gamma & , \text{ if } x_i < -\gamma \\ 0 & , \ x_i \in [-\gamma, \gamma] \end{cases}$$

Shrinkage

$$f(x) = \frac{\lambda}{2} \| Ax - b \|^2 \qquad g(x) = \|x\|_1$$

$$\nabla f(x) = \lambda A^T (Ax - b)$$

$$A(x) = \nabla f(x), \quad B(x) = \partial g(x)$$

$$x_{k+1} = (I + \gamma B)^{-1} (I - \gamma A)(x_k)$$

$$= T_\gamma [x_k - \gamma \lambda A^T (Ax_k - b)]$$

Forward-Backward splitting

- Projected Gradient
- Proximal Gradient
- IMEX
- ISTA