TV-minimization

$K^*K = -\Delta$

KK^*

$KU \approx \nabla u$

$K^*\vec{v} \approx -\nabla\cdot\vec{v}$

$$U \in \mathbb{R}^{n\times n}$$

$$K : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n} \otimes \mathbb{R}^{n\times n}$$

$$U \longmapsto \left(\frac{1}{h}UD^T, \frac{1}{h}DU\right)$$

$$\langle KU, \vec{v}\rangle = \langle \frac{1}{h}UD^T, V_1\rangle + \langle \frac{1}{h}DU, V_2\rangle$$

$$= \langle U, K^*\vec{v}\rangle$$

$$K^* : \mathbb{R}^{n\times n} \otimes \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$

$$\vec{v} \longmapsto \frac{1}{h}V_1 D + \frac{1}{h}D^T V_2$$

Given a noisy image $B \in \mathbb{R}^{n\times n}$, want to solve

$$\min_{U\in\mathbb{R}^{n\times n}} \|U\|_{TV} + \frac{\lambda}{2h}\|U-B\|_{L^2}^2$$

$$\|U-B\|_{L^2}^2 = \sum_i\sum_j h^2\cdot(U_{ij}-B_{ij})^2$$

$\lambda = 10\sim 15$ is usually good for images

$$\Longleftrightarrow \min_U \sum_i\sum_j\left[\sqrt{(DU)_{ij}^2 + (UD^T)_{ij}^2} + \frac{\lambda}{2}|U_{ij}-B_{ij}|^2\right]h$$

$$\Longleftrightarrow \min_U f(KU) + g(U) \qquad (P)$$

$$KU = (UD^T, DU)$$

$$g(U) = \frac{\lambda}{2}\sum_i\sum_j |U_{ij}-B_{ij}|^2$$

$$f(X,Y) = \sum_i\sum_j \sqrt{X_{ij}^2 + Y_{ij}^2}$$

$$KU = (X,Y)$$

$$\min_{U\in\mathbb{R}^{n\times n}} f(KU) + g(U) \qquad (P)$$

$$\min_{U} \max_{\vec{v}\in[\mathbb{R}^{n\times n}]^2} \langle \vec{v}, KU\rangle - f^*(\vec{v}) + g(U) \qquad (PD)$$

$$-\min_{\vec{v}} f^*(\vec{v}) + g^*(-K^*\vec{v}) \qquad (D)$$

PDHG is

$$\begin{cases} x_{k+1} = (I+\eta\partial g)^{-1}[x_k - \eta K^* y_k] \\ y_{k+1} = (I+\tau\partial f^*)^{-1}[y_k + \tau K(2x_{k+1}-x_k)] \end{cases}$$

$$\begin{cases} U_{k+1} = (I+\eta\partial g)^{-1}[U_k - \eta K^*\vec{v}_k] \\ \vec{v}_{k+1} = (I+\tau\partial f^*)^{-1}[\vec{v}_k + \tau K(2U_{k+1}-U_k)] \end{cases}$$

Remark ① If $K=I$, $\tau=\frac{1}{\eta}$, PDHG on (P)

$\iff$ Douglas-Rachford on (P)

② Even for $K\approx\nabla$, ADMM on (P) K^*K

$\iff$ Douglas-Rachford on (D)

Convergence of PDHG:

Proof I:

$$\begin{cases} x_{k+1} = (I+\eta\partial g)^{-1}[x_k - \eta K^* y_k] \\ y_{k+1} = (I+\tau\partial f^*)^{-1}[y_k + \tau K(2x_{k+1}-x_k)] \end{cases} \qquad \eta>0, \tau>0$$

$\eta\tau < \frac{1}{\rho(K^*K)}$, no need of strong convexity

The saddle point of $L(x,y) = \langle y, Kx\rangle - f^*(y) + g(x)$

satisfies

$$\begin{cases} 0 \in \frac{\partial L}{\partial x}(x_*, y_*) = K^* y_* + \partial g(x_*) \\ 0 \in -\frac{\partial L}{\partial y}(x_*, y_*) = -K x_* + \partial f^*(x_*) \end{cases}$$

$$0 \in T\begin{pmatrix} x_* \\ y_* \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} K^* y + \partial g(x) \\ -Kx + \partial f^*(y) \end{pmatrix} = \begin{pmatrix} \partial g(x) \\ \partial f^*(y) \end{pmatrix} + \begin{pmatrix} 0 & K^* \\ -K & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$0 \in T\begin{pmatrix} x_* \\ y_* \end{pmatrix} \qquad \vec{z} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad M = \begin{pmatrix} \frac{1}{\eta}I & -K^* \\ -K & \frac{1}{\tau}I \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} K^* y + \partial g(x) \\ -Kx + \partial f^*(y) \end{pmatrix} = \begin{pmatrix} \partial g(x) \\ \partial f^*(y) \end{pmatrix} + \begin{pmatrix} 0 & K^* \\ -K & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$0 \in M(\vec{z}_{k+1} - \vec{z}_k) + T\vec{z}_{k+1}$$

$$\Leftrightarrow \begin{cases} 0 \in \frac{1}{\eta}(x_{k+1} - x_k) - K^*(y_{k+1} - y_k) + \partial g(x_{k+1}) + K^* y_{k+1} \\ 0 \in \frac{1}{\tau}(y_{k+1} - y_k) - K(x_{k+1} - x_k) + \partial f^*(y_{k+1}) - K x_{k+1} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_k - \eta K^* y_k \in x_{k+1} + \eta\partial g(x_{k+1}) \\ y_k + \tau K(2x_{k+1} - x_k) \in y_{k+1} + \tau f^*(y_{k+1}) \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_{k+1} = (I+\eta\partial g)^{-1}[x_k - \eta K^* y_k] \\ y_{k+1} = (I+\tau\partial f^*)^{-1}[y_k + \tau K(2x_{k+1} - x_k)] \end{cases}$$

$$\vec{z} = \begin{pmatrix} x \\ y \end{pmatrix} \quad M = \begin{pmatrix} \frac{1}{\eta} I & -K^* \\ -K & \frac{1}{\tau} I \end{pmatrix}$$

So PDHG $\Longleftrightarrow \; M z_{k+1} + T z_{k+1} \ni M z_k$

$$\Longleftrightarrow \; z_{k+1} + M^{-1} T z_{k+1} \ni z_k$$

$$\Longleftrightarrow \; z_{k+1} = (I + M^{-1} T)^{-1} z_k$$

M is positive definite $\Longleftrightarrow \eta\tau < 1/\|K\|^2$

$$|\lambda I - M| = \begin{vmatrix} (\lambda - \frac{1}{\eta}) I & +K^* \\ +K & (\lambda - \frac{1}{\tau}) I \end{vmatrix} \qquad \|K\|^2 = \rho(K^* K)$$

$$= (\lambda - \tfrac{1}{\eta})^n \left| (\lambda - \tfrac{1}{\tau}) I - (\lambda - \tfrac{1}{\eta})^{-1} K K^* \right|$$

$$\Rightarrow \quad \eta\tau < 1/\|K\|^2$$

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - C A^{-1} B)$$

Define $\|z\|_M^2 = z^T M z = (x \;\; y) \begin{pmatrix} \frac{1}{\eta} I & -K^* \\ -K & \frac{1}{\tau} I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Then can prove

$$\|z_{k+1} - z_*\|_M \le \|z_k - z_*\|_M - c\|z_{k+1} - z_k\|_M$$

$\hookrightarrow$ A good reading project (2012 paper)

Proof II: $\min_{x} \max_{y} \langle Kx, y\rangle - f^*(y) + g(x)$

primal-dual gap:

$$G(x,y) = \max_{\hat{y}} \left[\langle \hat{y}, Kx\rangle - f^*(\hat{y}) + g(x) \right] \quad \langle y, Kx\rangle - f^*(y) + g(x)$$

$$- \min_{\hat{x}} \left[\langle y, K\hat{x}\rangle - f^*(y) + g(\hat{x}) \right] \geq 0$$

$$\begin{cases} x_{k+1} = (I + \eta\partial g)^{-1} [x_k - \eta K^* y_k] \\ y_{k+1} = (I + \tau \partial f^*)^{-1} [y_k + \tau K(2x_{k+1} - x_k)] \end{cases}$$

$$\begin{cases} x_{k+1} + \eta \partial g(x_{k+1}) \ni x_k - \eta K^* y_k \\ y_{k+1} + \tau \partial f^*(y_{k+1}) \ni y_k + \tau K(2x_{k+1} - x_k) \end{cases}$$

$$\Rightarrow \begin{cases} \partial g(x_{k+1}) \ni \frac{x_k - x_{k+1}}{\eta} - K^* y_k \\ \partial f^*(y_{k+1}) \ni \frac{y_k - y_{k+1}}{\tau} + K(2x_{k+1} - x_k) \end{cases}$$

$$\Rightarrow \begin{cases} g(x) \geq g(x_{k+1}) + \langle \frac{x_k - x_{k+1}}{\eta} - K^* y_k, x - x_{k+1}\rangle \\ f^*(y) \geq f^*(y_{k+1}) + \langle \frac{y_k - y_{k+1}}{\tau} + K(2x_{k+1} - x_k), y - y_{k+1}\rangle \end{cases}$$

$$\Rightarrow \begin{cases} g(x) \geq g(x_{k+1}) + \langle \frac{x_k - x_{k+1}}{\eta}, x - x_{k+1}\rangle - \langle K^* y_k, x - x_{k+1}\rangle \\ f^*(y) \geq f^*(y_{k+1}) + \langle \frac{y_k - y_{k+1}}{\tau}, y - y_{k+1}\rangle + \langle K(2x_{k+1} - x_k), y - y_{k+1}\rangle \end{cases}$$

subgradient definition

Add two :

$$\frac{\|y-y_k\|^2}{2\tau}+\frac{\|x-x_k\|^2}{2\eta} \geq \underbrace{\left[\langle Kx_{k+1}, y\rangle - f^*(y) + g(x_{k+1})\right]}_{L(x_{k+1},y)}$$

$$-\underbrace{\left[\langle Kx, y_{k+1}\rangle - f^*(y_{k+1}) + g(x)\right]}_{-L(x,y_{k+1})}$$

$$+\frac{\|y-y_{k+1}\|^2}{2\tau}+\frac{\|x-x_{k+1}\|^2}{2\eta}+\frac{\|y_k-y_{k+1}\|^2}{2\tau}+\frac{\|x_k-x_{k+1}\|^2}{2\eta}$$

$$+\langle K(x_k-x_{k+1}), y_{k+1}-y\rangle \quad = \langle K(x-x_{k+1}), y_{k+1}-y\rangle$$

$$-\langle K(x_{k+1}-x), y_{k+1}-y_k\rangle \quad +\langle K(x_k-x), y_k-y\rangle$$

$$+\langle K(x_k-x), y_{k+1}-y_k\rangle$$

$$\Rightarrow \quad L(x_{k+1},y)-L(x,y_{k+1})$$

$$+\underbrace{\frac{\|y-y_{k+1}\|^2}{2\tau}+\frac{\|x-x_{k+1}\|^2}{2\eta}-\langle K(x-x_{k+1}), y-y_{k+1}\rangle}_{\geq 0}$$

$$+\underbrace{\frac{\|y_k-y_{k+1}\|^2}{2\tau}+\frac{\|x_k-x_{k+1}\|^2}{2\eta}-\langle K(x_{k+1}-x_k), y_{k+1}-y_k\rangle}_{\geq 0}$$

$$\leq \frac{\|y-y_k\|^2}{2\tau}+\frac{\|x-x_k\|^2}{2\eta}-\langle K(x-x_k), y-y_k\rangle$$

$$\frac{\|a-b\|^2}{2\tau}+\frac{\|c-d\|^2}{2\eta}-\langle K(c-d), a-b\rangle$$

$$\geq \frac{\|a-b\|^2}{2\tau}+\frac{\|c-d\|^2}{2\eta}-\|K(c-d)\|\cdot\|a-b\|$$

$$\geq \frac{\|a-b\|^2}{2\tau}+\frac{\|c-d\|^2}{2\eta}-\|K\|\cdot\|c-d\|\cdot\|a-b\| \geq 0 \text{ if } \tau\eta \leq \frac{1}{\|K\|^2}$$

$$\geq \frac{A^2}{2\tau} + \frac{B^2}{2\eta} - \tau\eta AB = \frac{1}{2}\left[\frac{A}{\tau} - \frac{B}{\eta}\right]^2 \geq 0$$

$$\Rightarrow \sum_{i=1}^{k} \left[L(x_i, y) - L(x, y_i)\right]$$

$$\underbrace{+ \frac{\|y - y_{k+1}\|^2}{2\tau} + \frac{\|x - x_{k+1}\|^2}{2\eta} - \langle K(x - x_{k+1}), y - y_{k+1}\rangle}_{\geq 0}$$

$$\leq \frac{1}{2\tau}\|x - x_0\|^2 + \frac{1}{2\sigma}\|y - y_0\|^2 - \langle K(x - x_0), (y - y_0)\rangle$$

$$\leq \frac{1}{\tau}\|x - x_0\|^2 + \frac{1}{\sigma}\|y - y_0\|^2$$

Let $\bar{x}_k = \frac{1}{k}\sum_{i=1}^{k} x_i$

$$L(\bar{x}_k, y) = \langle K\bar{x}_k, y\rangle - f^*(y) + g(\bar{x}_k)$$

Jensen's Inequali

$$\leq \frac{1}{k}\sum_{i=1}^{k}\left[\langle Kx_i, y\rangle - f^*(y) + g(x_i)\right]$$

$$-L(x, \bar{y}_k) = -\left[\langle Kx, \bar{y}_k\rangle - f^*(\bar{y}_k) + g(x)\right]$$

$$\leq -\frac{1}{k}\sum_{i=1}^{k}\left[\langle Kx, y_i\rangle - f^*(y_i) + g(x)\right]$$

$$\Rightarrow L(\bar{x}_k, y) - L(x, \bar{y}_k) \leq \frac{1}{k}\left[\frac{1}{\tau}\|x - x_0\|^2 + \frac{1}{\sigma}\|y - y_0\|^2\right]$$

$O(\frac{1}{k})$