Nesterov 1983: $\begin{cases} x_{k+1} = y_k - \eta_k \nabla f(y_k) \\ y_{k+1} = x_{k+1} + \frac{t_k - 1}{t_{k+1}}(x_{k+1} - x_k) \end{cases}$ $O(\frac{1}{k^2})$

Beck & Teboulle 2009: $\min_{x \in \mathbb{R}^n} f(x) + \|x\|_1$ $\quad \|Ax-b\|_2 + \lambda\|x\|_1$

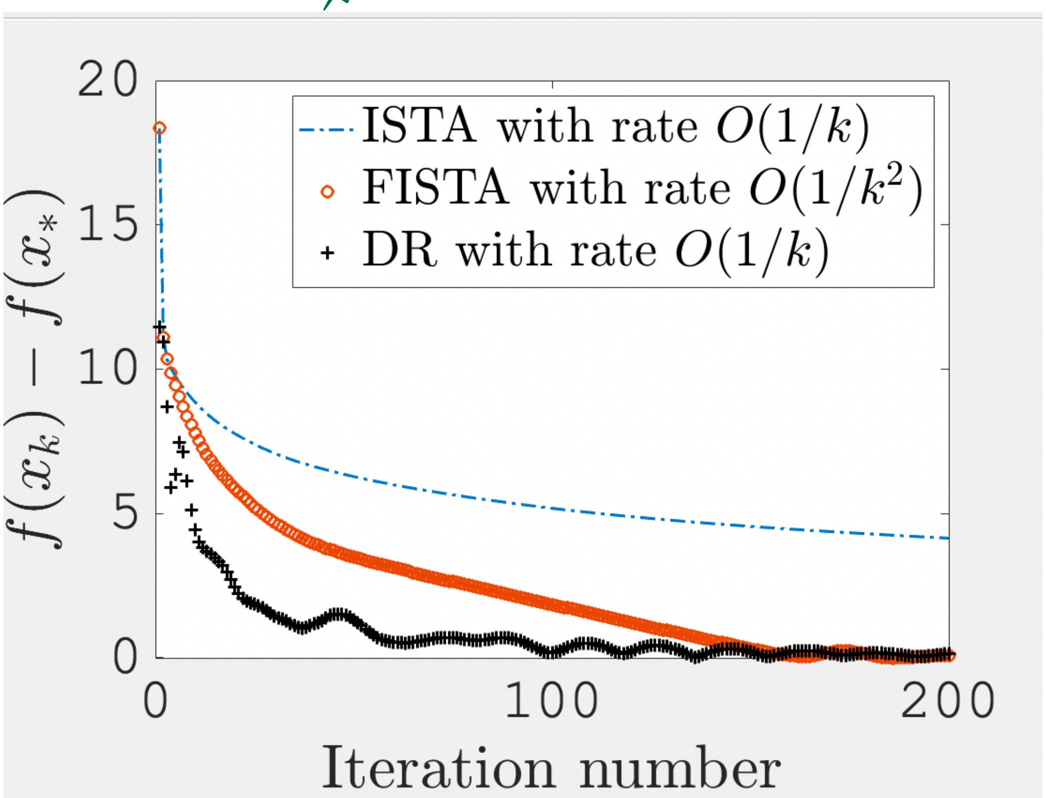
$O(\frac{1}{k^2})$

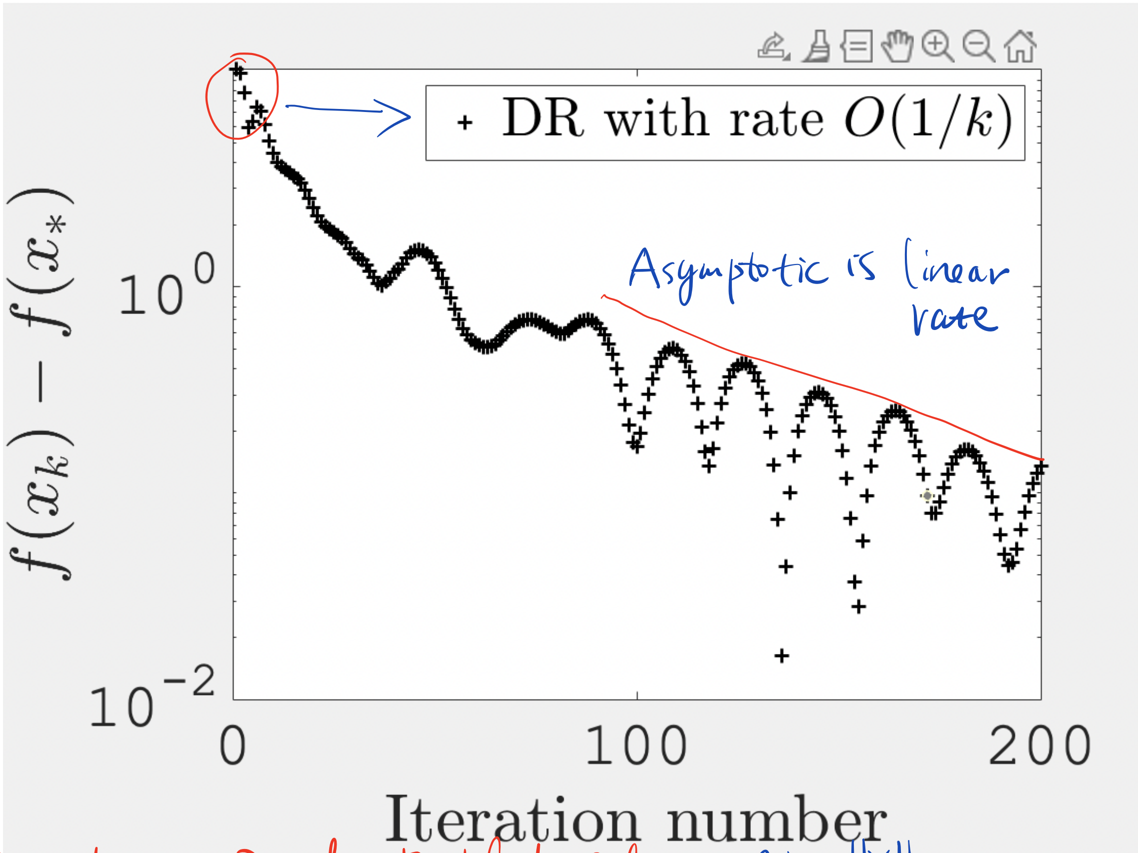
FISTA $\begin{cases} x_{k+1} = T_{\frac{1}{L}}[y_k - \frac{1}{L}\nabla f(y_k)] \\ y_{k+1} = x_{k+1} + \frac{t_k - 1}{t_{k+1}}(x_{k+1} - x_k) \end{cases}$

$$T_{\frac{1}{L}}(x)_i = \begin{cases} x_i, & \text{if } -\frac{1}{L} \le x_i \le \frac{1}{L} \\ x_i - \frac{1}{L}, & \text{if } x_i > \frac{1}{L} \\ x_i + \frac{1}{L}, & \text{if } x_i < -\frac{1}{L} \end{cases}$$

ISTA: $x_{k+1} = T_{\frac{1}{L}}[x_k - \frac{1}{L}\nabla f(x_k)]$ $\quad O(\frac{1}{k})$

$$\min_x \|Ax - b\|^2 + \lambda\|x\|_1$$





Remark: ① Douglas-Rachford solves $\min \|x\|_1$ s.t. $Ax = b$

② In each iteration of Douglas-Rachford, one needs to compute $(A^TA)^{-1}$.

$$\min_x f(x)$$

$$\begin{cases} x_{k+1} = y_k - \eta \nabla f(y_k) \\ y_{k+1} = x_{k+1} + \frac{t_k - 1}{t_{k+1}} (x_{k+1} - x_k) \end{cases}$$

$$f(x_{k+1}) \leq f(y_k) + \langle \nabla f(y_k), x_{k+1} - y_k \rangle + \frac{L}{2} \| y_k - x_{k+1} \|^2$$

$$f(x_k) \geq f(y_k) + \langle \nabla f(y_k), x_k - y_k \rangle$$

$$f(x_k)-f(x_{k+1}) \geq -\frac{L}{2}\|y_k-x_{k+1}\|^2+\langle \nabla f(y_k), x_k-x_{k+1}\rangle$$

$$= -\frac{L}{2}\|y_k-x_{k+1}\|^2+\langle \frac{1}{\eta}(y_k-x_{k+1}), x_k-x_{k+1}\rangle$$

$$= -\frac{L}{2}\|y_k-x_{k+1}\|^2+\langle \frac{1}{\eta}(y_k-x_{k+1}), x_k-y_k+y_k-x_{k+1}\rangle$$

$$= \left(\frac{1}{\eta}-\frac{L}{2}\right)\|y_k-x_{k+1}\|^2+\frac{1}{\eta}\langle y_k-x_{k+1}, x_k-y_k\rangle$$

$$\eta=\frac{1}{L} \Leftrightarrow \frac{1}{\eta}-\frac{L}{2}=\frac{L}{2}, \quad \frac{1}{\eta}=L$$

① $f(x_k)-f(x_{k+1}) \geq \frac{L}{2}\|y_k-x_{k+1}\|^2+L\langle y_k-x_{k+1}, x_k-y_k\rangle$

$$\begin{cases} f(x_{k+1}) \leq f(y_k)+\langle \nabla f(y_k), x_{k+1}-y_k\rangle+\frac{L}{2}\|y_k-x_{k+1}\|^2 \\ f(x_*) \geq f(y_k)+\langle \nabla f(y_k), x_*-y_k\rangle \end{cases}$$

$$f(x_*)-f(x_{k+1}) \geq -\frac{L}{2}\|y_k-x_{k+1}\|^2+\langle \nabla f(y_k), x_*-x_{k+1}\rangle$$

$$= -\frac{L}{2}\|y_k-x_{k+1}\|^2+\langle \frac{y_k-x_{k+1}}{\eta}, x_*-y_k+y_k-x_{k+1}\rangle$$

$$\eta=\frac{1}{L}$$

$\Rightarrow$

② $f(x_*)-f(x_{k+1}) \geq \frac{L}{2}\|y_k-x_{k+1}\|^2+L\langle y_k-x_{k+1}, x_*-y_k\rangle$

① $f(x_k)-f(x_{k+1}) \geq \frac{L}{2}\|y_k-x_{k+1}\|^2+L\langle y_k-x_{k+1}, x_k-y_k\rangle$

$$R_k=f(x_k)-f(x_*)$$

$$R_k - R_{k+1} = f(x_k) - f(x_{k+1})$$

$$(t_k - 1)\cdot ① + ② \Rightarrow$$

$$(t_k-1)[R_k - R_{k+1}] - R_{k+1} \geq \frac{L}{2} t_k \|y_k - x_{k+1}\|^2$$

$$+ L \langle y_k - x_{k+1}, (t_k-1)x_k - t_k y_k - x_* \rangle$$

$$\underline{t_k(t_k-1)}R_k - t_k^2 R_{k+1} \geq \frac{L}{2}\|t_k(y_k - x_{k+1})\|^2$$

$$\leq t_{k-1}^2$$

$$+ L \langle t_k(y_k - x_{k+1}), (t_k-1)x_k - t_k y_k - x_* \rangle$$

$$t_k^2 - t_k \leq t_{k-1}^2$$

$$\Rightarrow t_{k-1}^2 R_k - t_k^2 R_{k+1} \geq \frac{L}{2}\|t_k(y_k - x_{k+1})\|^2$$

$$+ L \langle t_k(y_k - x_{k+1}), \underbrace{(t_k-1)x_k - x_*}_{c} - \underbrace{t_k y_k}_{a} \rangle$$

$$b = t_k x_{k+1}$$

$$RHS = \frac{L}{2}\|a-b\|^2 + L\langle a-b, c-a \rangle$$

$$= \frac{L}{2}\left(\|a\|^2 + \|b\|^2 - 2a\cdot b - 2\|a\|^2 + 2a\cdot b + 2a\cdot c - 2b\cdot c\right)$$

$$= \frac{L}{2}\left(\|b\|^2 - 2b\cdot c + \|c\|^2 - \|c\|^2 + 2a\cdot c - \|a\|^2\right)$$

$$= \frac{L}{2}\left(\|b-c\|^2 - \|a-c\|^2\right)$$

$$\Rightarrow t_{k-1}^2 R_k - t_k^2 R_{k+1} \geq \frac{L}{2}\left(\|u_{k+1}\|^2 - \|u_k\|^2\right)$$

$$u_{k+1} = b - c = t_k x_{k+1} - [(t_k-1)x_k + x_*]$$

$$\Rightarrow \quad t_{k-1}^2 R_k + \frac{L}{2}\|u_k\|^2 \geq t_k^2 R_{k+1} + \frac{L}{2}\|u_{k+1}\|^2$$

$$\Rightarrow \quad t_k^2 R_{k+1} + \frac{L}{2}\|u_{k+1}\|^2 \leq t_0^2 R_1 + \frac{L}{2}\|u_1\|^2$$

$$\Rightarrow \quad t_k^2 R_{k+1} \leq t_0^2 R_1 + \frac{L}{2}\|u_1\|^2$$

$$\Rightarrow \quad R_{k+1} \leq \frac{1}{t_k^2}\left[t_0^2 R_1 + \frac{L}{2}\|u_1\|^2 \right]$$

$$\left(\left.\begin{matrix} t_k^2 - t_k \leq t_{k-1} \\ t_0 = 1 \end{matrix}\right\} \Rightarrow t_k \geq \frac{k+1}{2} \right)$$

$$\Rightarrow \quad R_{k+1} \leq \frac{4}{(k+1)^2}\left[t_0^2 R_1 + \frac{L}{2}\|u_1\|^2 \right]$$