Computing Resonances by Generalized Complex Scaling

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We are interested in the resonances of the Hamiltonian $H_V = D^2 + V(x)$, where $V \in L^\infty_{\text{comp}}(\mathbb{R})$ (bounded compactly supported functions) and $D = -i\partial_x$. A simple resonance is a $\lambda \in \mathbb{C}\backslash 0$ for which there is a purely outgoing state, i.e. a solution ϕ to

$$(H_V - \lambda^2)\phi = 0$$

satisfying

$$\phi(x) = \begin{cases} Ae^{i\lambda x} & x \geq R \\ Be^{-i\lambda x} & x \leq -R \end{cases}$$

A resonance of multiplicity n is a $\lambda \in \mathbb{C}\backslash 0$ for which there is a ϕ such that $(H_V - \lambda^2)^{n-1}\phi$ is an outgoing state.

As an aid in the analysis of $D^2 + V(x)$ we will use a curve $\Gamma \subset \mathbb{C}$ of the following type: Suppose $\text{supp}(V) \subset (-R, R)$, and let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function satisfying $g([-R, R]) = 0$. We will consider curves $\Gamma \subset \mathbb{C}$ which are graphs of such functions g.

We now define differentiation and integration of functions mapping Γ to $\mathbb{C}$. Let $\gamma(t)$ be a parametrization $\mathbb{R} \to \Gamma$, and let $f \in C^1(\Gamma)$ in the sense that $f \circ \gamma \in C^1(\mathbb{R})$. We define

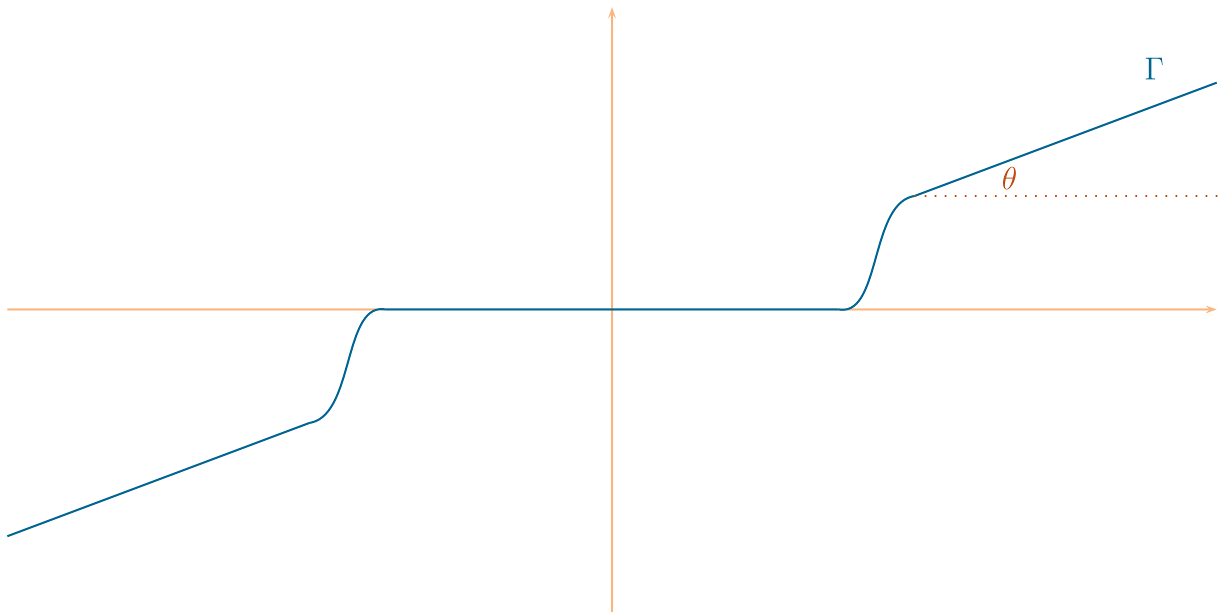
$$\partial_z^\Gamma f(z_0) = \gamma'(t_0)^{-1}\partial_t(f \circ \gamma)(t_0)$$

where $\gamma(t_0) = z_0$ (the inverse and the multiplication are in the sense of complex numbers), and further define $D_z = -i\partial_z^\Gamma$. By the chain rule, if f is differentiable in a neighborhood of Γ, this is independent of parametrization. In fact, if $\gamma = \gamma_1 + i\gamma_2$, then

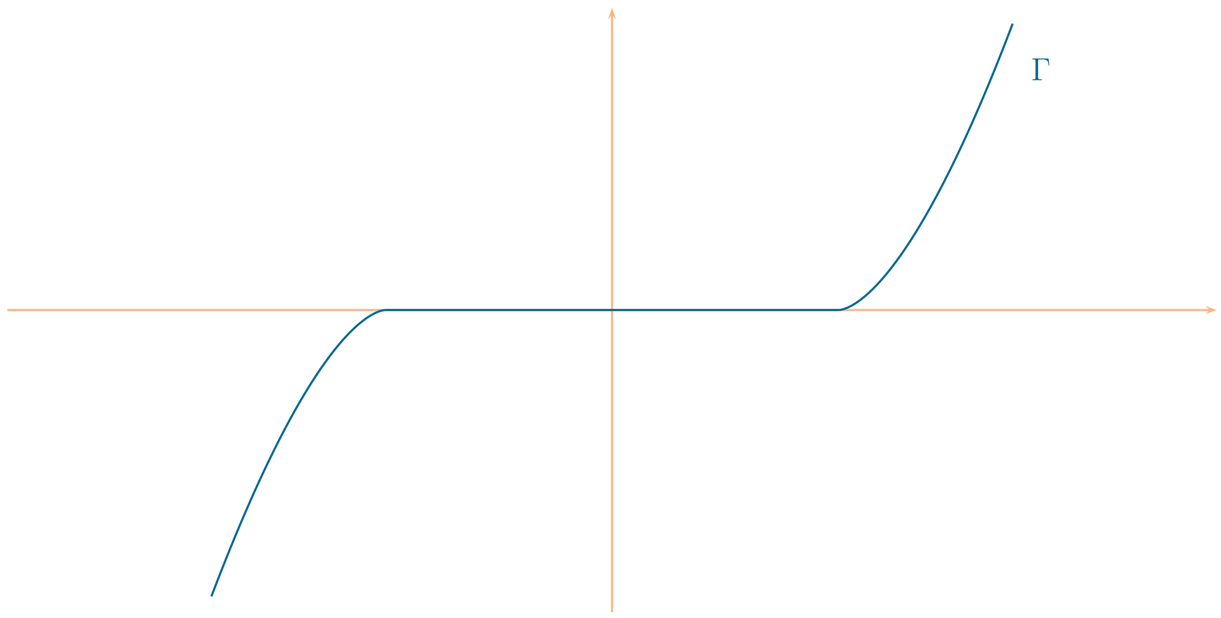
$$\gamma'(t_0)^{-1}\partial_t(f \circ \gamma)(t_0) = \gamma'(t_0)^{-1}(\partial_x f(z_0)\gamma_1'(t_0) + \partial_y f(z_0)\gamma_2'(t_0))$$

If α is another parametrization passing through z_0 at time s_0, then $\gamma'(t_0) = c(s_0)\alpha'(s_0)$ for some factor $c(s_0) \in \mathbb{R}$, because the two tangent vectors must be parallel. This factor cancels in the expression above, guaranteeing independence of parametrization. Moreover, if f is holomorphic in a neighborhood of Γ, then $\partial_y f = i\partial_x f$ and the entire expression reduces to $\partial_z^\Gamma f = \partial_x f = \partial_z f$, so in this case ∂_z^Γ coincides with the holomorphic differential operator. To integrate along the curve we can use both the complex contour measure, denoted $dz = \gamma'(t)dt$, and the arclength measure, denoted $d|z| = |\gamma'(t)|dt$. The space $L^2(\Gamma)$ is defined using the second measure.

Note that V is a well defined function on Γ, so that putting $H_\Gamma \overset{def}{=} (D_z^\Gamma)^2 + V(z)$ makes sense. We wish to argue that certain resonances of H_V coincide with eigenvalues of H_Γ as a (densely defined) operator on $L^2(\Gamma)$, in agreement with multiplicity. This reformulates the problem of computing resonances as a search for eigenvalues. This more general setting includes both the usual complex scaling as well its modified form as it is used in Perfectly Matched Layers (PML) computations[1].

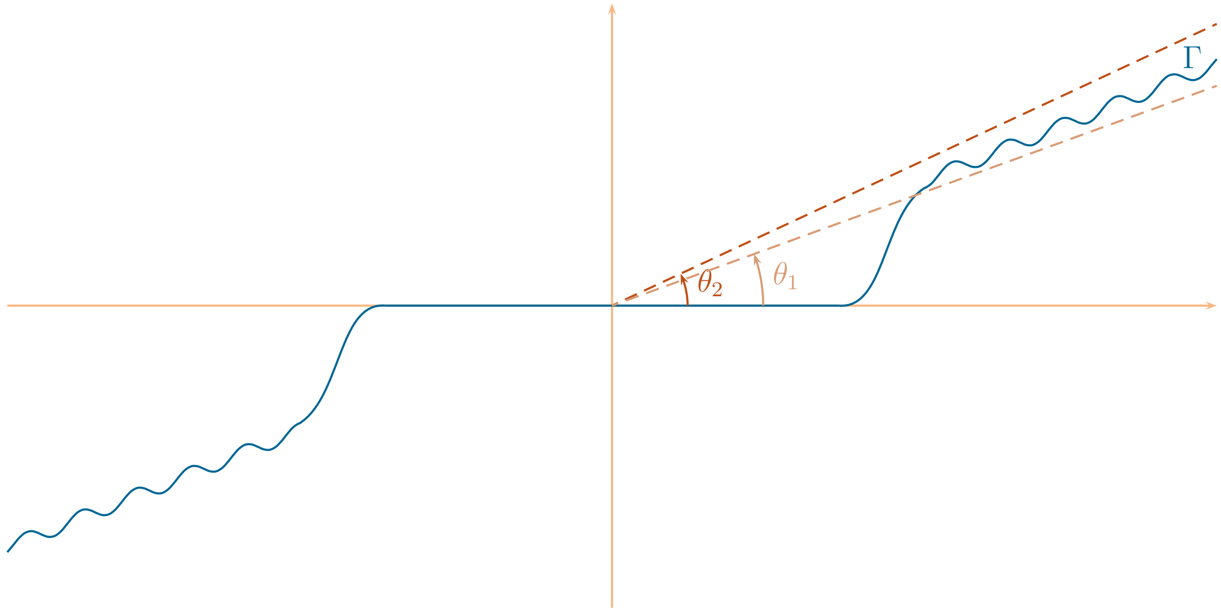


Curve Γ used in complex scaling. The curve is given by a C^∞ function g satisfying $g(x) = 0$ for $-R \leq x \leq R$ and $g(x) = x \tan\theta$ for $|x|$ sufficiently large, where θ is a given constant.



Curve Γ used in PML computations. A typical curve is given by a function g satisfying $g(x) = -|x+R|^\alpha$ for $x < -R$, $g(x) = 0$ for $-R \leq x \leq R$, and $g(x) = (x-R)^\alpha$ for $x > R$, where $\alpha > 1$.

For the main theorem we introduce two angles as follows. Fix $0 \leq \theta_1 \leq \theta_2 \leq \pi/2$ and $\varepsilon > 0$ having the property that for every $z \in \Gamma$ outside of some compact set, either $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$ or $\theta_1 + \varepsilon \leq \arg(-z) \leq \theta_2 - \varepsilon$. In the first case above we may take $\theta_1 = \theta - \varepsilon$ and $\theta_2 = \theta + \varepsilon$. In the second case above we may take $\theta_1 = \pi/2 - \varepsilon$ and $\theta_2 = \pi/2$. The following graph illustrates how θ_1 and θ_2 may be chosen:



Theorem. *Any λ with $-\theta_1 \leq \arg \lambda \leq \pi - \theta_2$ is an eigenvalue of H_Γ of multiplicity m if and only if it is a resonance of H_V of multiplicity m.*

Proof. Suppose λ is a resonance of multiplicity m of H_V. This means that there is a function $\phi : \mathbb{R} \to \mathbb{C}$ satisfying $(H_V - \lambda^2)^m \phi(x) = 0$ and

$$(H_V - \lambda^2)^{m-1}\phi(x) = \begin{cases} Ae^{i\lambda x} & x \geq R \\ Be^{-i\lambda x} & x \leq -R \end{cases}$$

This means that ϕ satisfies

$$\phi(x) = \begin{cases} P(x)e^{i\lambda x} & x \geq R \\ Q(x)e^{-i\lambda x} & x \leq -R \end{cases}$$

for suitable polynomials P and Q. We now define $\tilde{\phi} : \Gamma \to \mathbb{C}$ as follows

$$\tilde{\phi}(z) = \begin{cases} P(z)e^{i\lambda z} & \mathrm{Re}(z) \geq R \\ \phi(z) & -R < \mathrm{Re}(z) < R \\ Q(z)e^{-i\lambda z} & \mathrm{Re}(z) \leq -R \end{cases}$$

This $\tilde{\phi}$ clearly satisfies $(H_\Gamma - \lambda^2)^m \tilde{\phi} = 0$ and $(H_\Gamma - \lambda^2)^{m-1}\tilde{\phi} \neq 0$. If now $\tilde{\phi} \in L^2(\Gamma)$ this will imply that H_Γ has an eigenvalue of multiplicity at least m at λ. We defer the proof of this lemma momentarily. This means that if λ is a resonance of multiplicity m for H_V, then it is an eigenvalue of multiplicity at least m for H_Γ. In the same way one shows that if λ is an eigenvalue of multiplicity m for H_Γ, then it is a resonance of multiplicity m for H_V. Observe that in this one-dimensional problem the only possible multiplicities are algebraic. Geometric multiplicites are ruled out by the rigid form of the eigenfunctions of H_Γ for $|\operatorname{Re}(z)| \geq R$. $\square$

To complete the proof we need only the following lemma.

Lemma. *Any continuous function $f : \Gamma \to \mathbb{C}$ satisfying*

$$f(z) = \begin{cases} P(z)e^{i\lambda z} & \operatorname{Re}(z) \geq R \\ Q(z)e^{-i\lambda z} & \operatorname{Re}(z) \leq -R \end{cases}$$

where $P(z)$ and $Q(z)$ are polynomials, is in $L^2(\Gamma)$.

Proof. We must show that

$$\int_\Gamma |f(z)|^2 d|z| < \infty$$

To prove this it is enough to consider the tail of the integral, where $\theta_1 \leq \arg z \leq \theta_2$. Suppose this inequality holds for $\operatorname{Re}(z) \geq T$, and that $T \geq R$, and let $\Gamma_T = \Gamma \cap \{\operatorname{Re}(z) \geq T\}$. Then

$$\begin{aligned} \int_{\Gamma_T} |f(z)|^2 d|z| &= \int_{\Gamma_T} |P(z)||e^{i\lambda z}|^2 d|z| = \int_{\Gamma_T} |P(z)| e^{-2(\operatorname{Im}(\lambda)\operatorname{Re}(z) + \operatorname{Re}(\lambda)\operatorname{Im}(z))} d|z| \\ &= \int_{\Gamma_T} |P(z)| e^{-2|z||\lambda|\sin(\arg\lambda + \arg z)} d|z| \leq \int_{\Gamma_T} |P(z)| e^{-2|z||\lambda|\sin\varepsilon} d|z| \\ &= \int_T^\infty |P(t+ig(t))| e^{-C\sqrt{t^2+g(t)^2}} \sqrt{1+g'(t)^2}\, dt \end{aligned}$$

We now reduce to the case $P(z) = 1$. For $|z|$ sufficiently large, $|P(z)| \leq A|z|^n$, and, for a suitable constant C', $A|z|^n e^{-C|z|} \leq e^{-C'z}$ when $|z|$ is sufficiently large. If now $P(z)$ is not 1, we reduce to this case by replacing C by C' and T by T' where T' is large enough that this last inequality holds. We then write $[T, \infty) = A \cup B$, where $A = \{t \geq T : 1 \geq g'(t)\}$ and $B = \{t \geq T : 1 \leq g'(t)\}$. This allows us to divide our integral into two parts which we bound separately:

$$\int_A e^{-C\sqrt{t^2+g(t)^2}} \sqrt{1+g'(t)^2}\, dt \leq \sqrt{2} \int_A e^{-Ct} dt < \infty$$

On B we use the fact that g is monotone to effect a change of variables:

$$\int_B e^{-C\sqrt{t^2+g(t)^2}} \sqrt{1+g'(t)^2}\, dt \leq \sqrt{2} \int_T^\infty e^{-Cg(t)} g'(t) dt = \sqrt{2} \int_{g(T)}^{g(\infty)} e^{-Cu} du < \infty$$

The proof of convergence of the other side of the integral follows the same line of reasoning. $\square$

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References

[1] J. P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves", J. Comp. Phy., vol. 114, pp. 185-200, Oct. 1994.
[2] J. Sjöstrand and M. Zworski, "Complex scaling and the distribution of scattering poles", J. Amer. Math. Soc., vol 4 , no. 4, pp. 729-769, 1991.