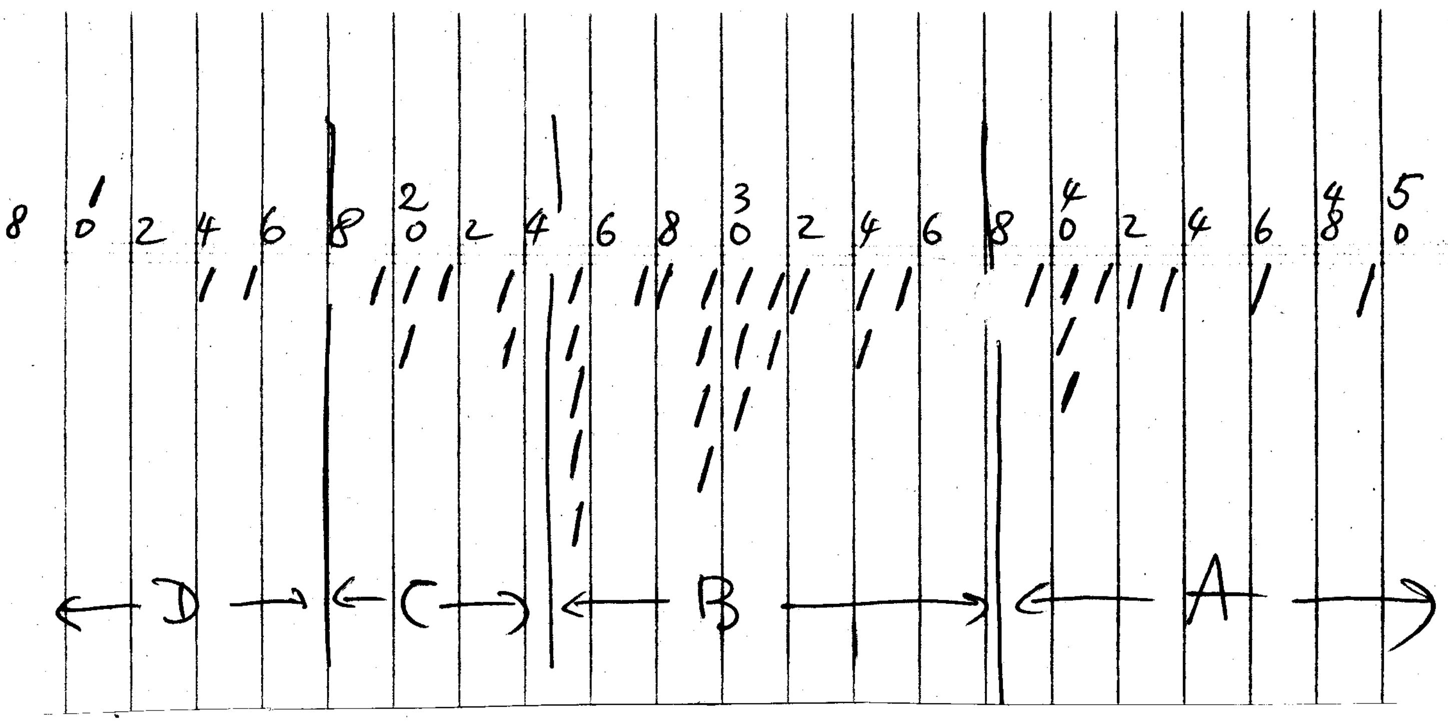
Purdue University MA 173
Calculus and Analytic Geometry II
Fall 2003, Test One

(Instructor: Aaron N. K. Yip)

- This test booklet has FIVE QUESTIONS, totaling 50 points for the whole test. You have 50 minutes to do this test. **Plan your time well. Read the questions carefully. You do not need to attempt the questions in sequence.**
- This test is **closed book and note. No calculator** is allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- You can use both sides of the papers to write your answers. But please indicate so if you do.



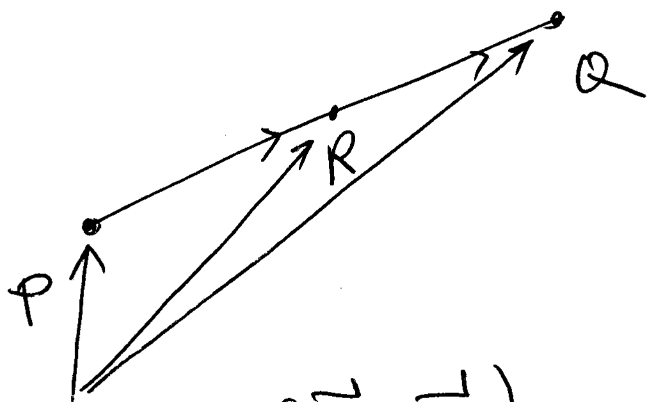
1. Consider the straight line segment PQ with end points P and Q. Let R be the point on the segment PQ such that

$$\boxed{\frac{|PR|}{|RQ|} = \frac{\alpha}{\beta}} \text{ where } |\cdot| \text{ denotes the length of a vector.}$$

Express the vector $\vec{R}$ in terms of $\vec{P}$, $\vec{Q}$, α and β.

(Hint: What is the relationship between $\vec{PR}$ and $\vec{RQ}$?)

$$\beta |\overrightarrow{PR}| = \alpha |\overrightarrow{RQ}|$$



$$\beta \overrightarrow{PR} = \alpha \overrightarrow{RQ}$$

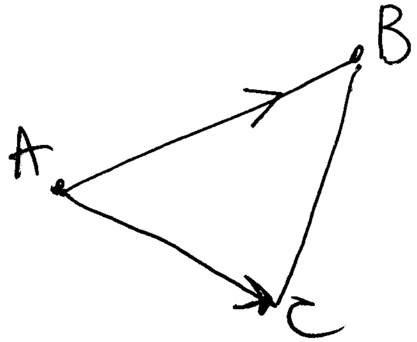
$$\beta(\vec{R} - \vec{P}) = \alpha(\vec{Q} - \vec{R})$$

$$\beta \vec{R} - \beta \vec{P} = \alpha \vec{Q} - \alpha \vec{R}$$

$$(\alpha + \beta)\vec{R} = \alpha \vec{Q} + \beta \vec{P}$$

$$\boxed{\vec{R} = \frac{\alpha \vec{Q} + \beta \vec{P}}{\alpha + \beta}}$$

2. Given the triangle $\triangle ABC$ in R^3 with: $A = (1,2,-1)$, $B = (1,1,1)$ and $C = (-2,1,4)$. Find the cosine of the angle at A and also the area of $\triangle ABC$.



$\overrightarrow{AB} = (0,-1,2)$, $\quad |\overrightarrow{AB}| = \sqrt{1^2+2^2} = \sqrt{5}$

$\overrightarrow{AC} = (-3,-1,5) \quad |\overrightarrow{AC}| = \sqrt{9+1+25} = \sqrt{35}$

$$\cos\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|AB|\,|AC|} = \frac{1+10}{\sqrt{5}\sqrt{35}} = \frac{11}{(\sqrt{7})5}.$$

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{vmatrix} i & j & k \\ 0 & -1 & 2 \\ -3 & -1 & 5 \end{vmatrix}\right|$$

$$\begin{array}{r} 36 \\ 18 \\ \hline 54 \end{array}$$

$$= \frac{1}{2}\left[i\begin{pmatrix} -1 & 2 \\ -1 & 5 \end{pmatrix} - j\begin{pmatrix} 0 & 2 \\ -3 & 5 \end{pmatrix} + k\begin{pmatrix} 0 & -1 \\ -3 & -1 \end{pmatrix}\right]$$

$$= \left|\frac{1}{2}\left[-3\hat{i} - 6\hat{j} - 3\hat{k}\right]\right| \qquad \text{Area} = \frac{1}{2}\sqrt{9+36+9}$$

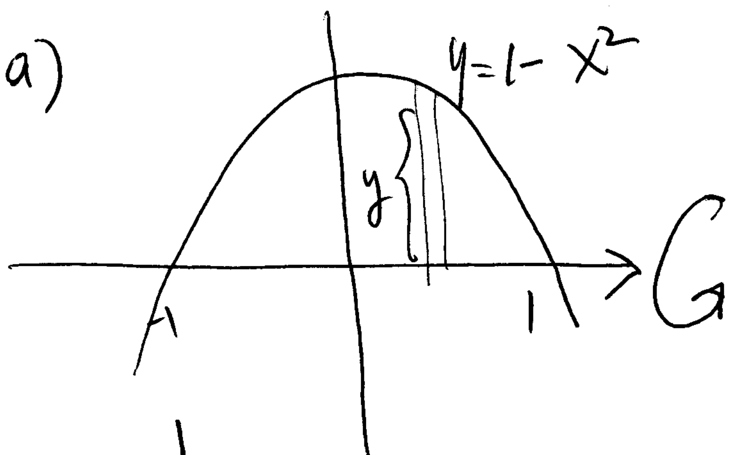
$$= \frac{1}{2}\sqrt{54} = \frac{3}{2}\sqrt{6} \quad \text{※}$$

4. Consider the region under the graph $y = 1 - x^2$ and above the x-axis. Find the volume of the solid formed by

(a) rotating the region with respect to the x-axis.

(b) rotating the region with respect to the y-axis.

(a)



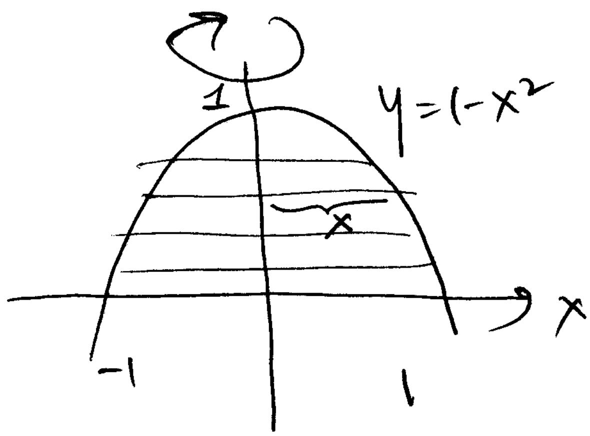
$$V = \int_{-1}^{1} \pi y^2 \, dx$$

$$= \pi \int_{-1}^{1} (1-x^2)^2 \, dx$$

$$= 2\pi \int_0^1 (1 - 2x^2 + x^4) \, dx = 2\pi \left[1 - \frac{2}{3} + \frac{1}{5} \right] = 2\pi \left[\frac{15 - 10 + 3}{15} \right]$$

$$= \frac{16\pi}{15} =$$

(b)



$$V = \int_0^1 \pi x^2 \, dy = \int_0^1 \pi (1 - y) \, dy = \pi \left[1 - \frac{1}{2} \right]$$

$$= \frac{\pi}{2}$$

5. Compute the following integrals:

$$\text{(a)} \quad \int_0^1 x\sqrt{1-x^2}\,dx$$
$$\text{(b)} \quad \int_0^{\frac{3}{2}} \sqrt{9-4x^2}\,dx$$
$$\text{(c)} \quad \int_{-4}^4 y\sqrt{16-y^2}\,dy$$
$$\text{(d)} \quad \int_{-4}^4 \sqrt{16-y^2}\,dy$$

(Hint: You can use the fact that: $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$.)

$$\text{(a)} \quad \int_0^1 x\sqrt{1-x^2}\,dx = \int_0^1 \sqrt{1-u}\,\frac{du}{2} \qquad u = x^2, \quad du = 2x\,dx$$

$$= \frac{1}{2}(1-u)^{3/2}\left(-\frac{2}{3}\right)\Big|_0^1 = 1/3 \#$$

$$\text{(b)} \quad \int_0^{3/2} \sqrt{9-4x^2}\,dx = \int_0^1 \sqrt{9-9u^2}\,\frac{3}{2}\,du$$

$u = \frac{2x}{3}$, $du = \frac{2}{3}dx$

$$\sqrt{9-4x^2} = \sqrt{9-4\left(\frac{3u}{2}\right)^2} = \sqrt{9-9u^2}$$

$$= \frac{9}{2}\underbrace{\int_0^1 \sqrt{1-u^2}\,du}_{\pi/4} = \frac{9\pi}{8} \#$$

$$\text{(c)} \quad \int_{-4}^4 y\sqrt{16-y^2}\,dy = 0$$

($\because$ $y\sqrt{16-y^2}$ is an odd function) integrate over an symmetric interval.

This is a scrap paper.

(d) $$\int_{-4}^{4} \overbrace{\sqrt{16-y^2}}^{\text{even fct}}\, dy = 2\int_0^4 \sqrt{16-y^2}\, dy$$

$u = \frac{y}{4}$, $y = 4u$

$dy = 4\,du$

$$= 2\int_0^1 \sqrt{16-16u^2}\; 4\, du$$

$$= 32\int_0^1 \sqrt{1-u^2}\, du$$

$$= 32\left(\pi/4\right)$$

$$= 8\pi \;\#$$