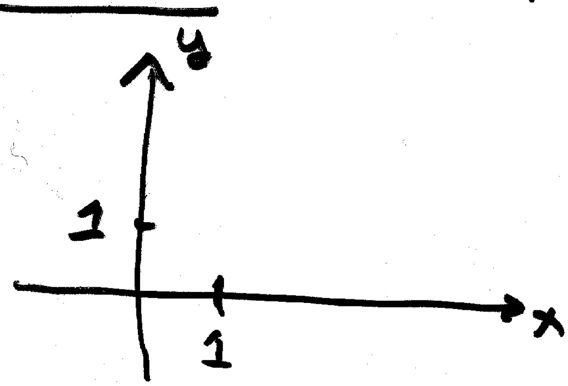
1

Three dimensional coordinate system

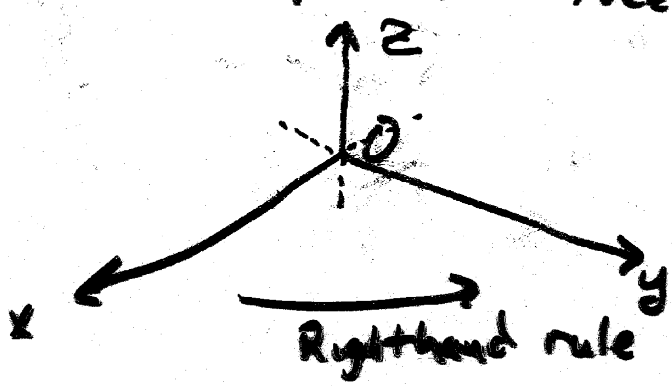
Just as, in the plane, we can locate any point by coordinates, we can assign coordinates in three dimensional space.

Recall <u>conventions</u> for plane coordinates:

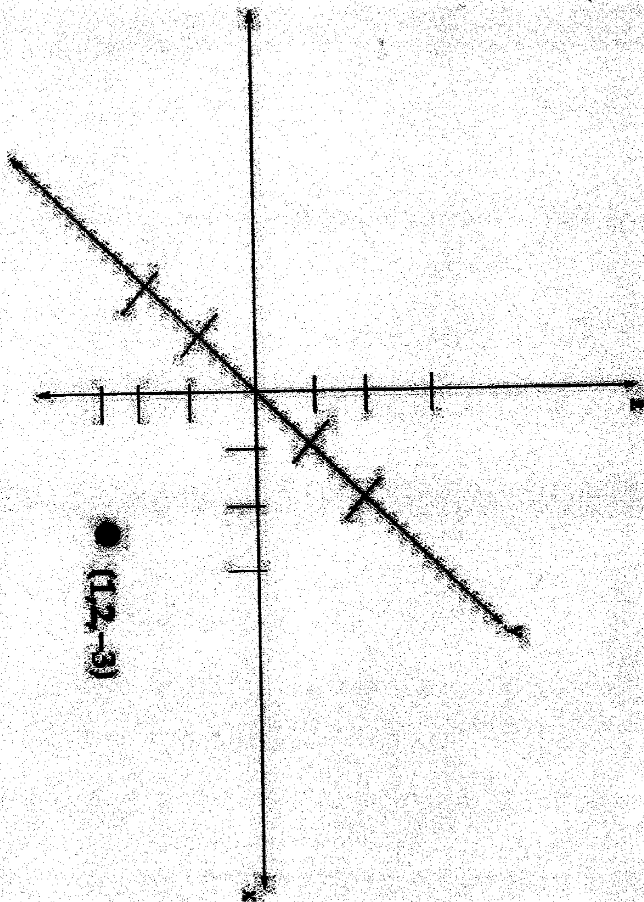


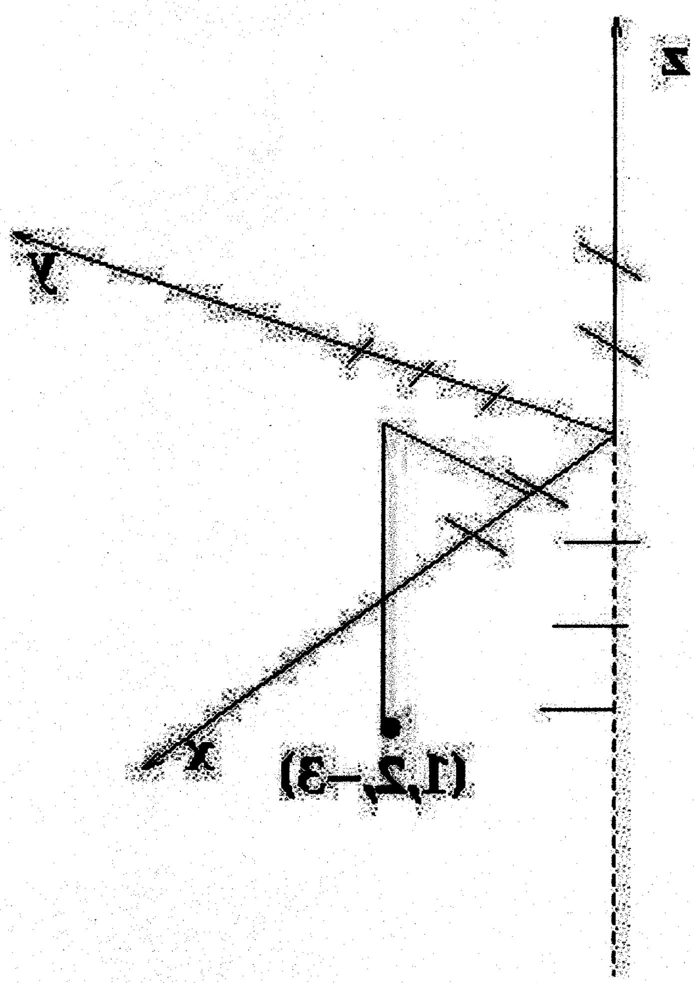
The positive directions are <u>standard</u>, as is the orientation x (first coordinate) horizontal
y (second coordinate) vertical.

3 dimensional space: Need convention:

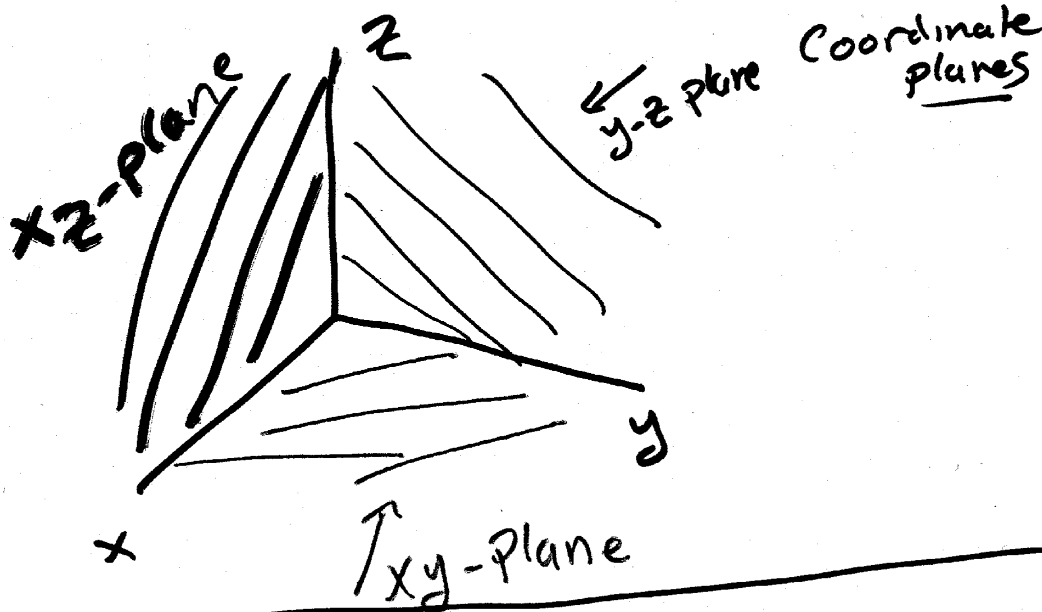


1a




z
y
x
(1,2,-3)

2



$\boxed{\text{We refer to the space of all points as } \mathbb{R}^3}$

To locate a point, we need 3 coordinate

(x, y, z) describing the distance to the point

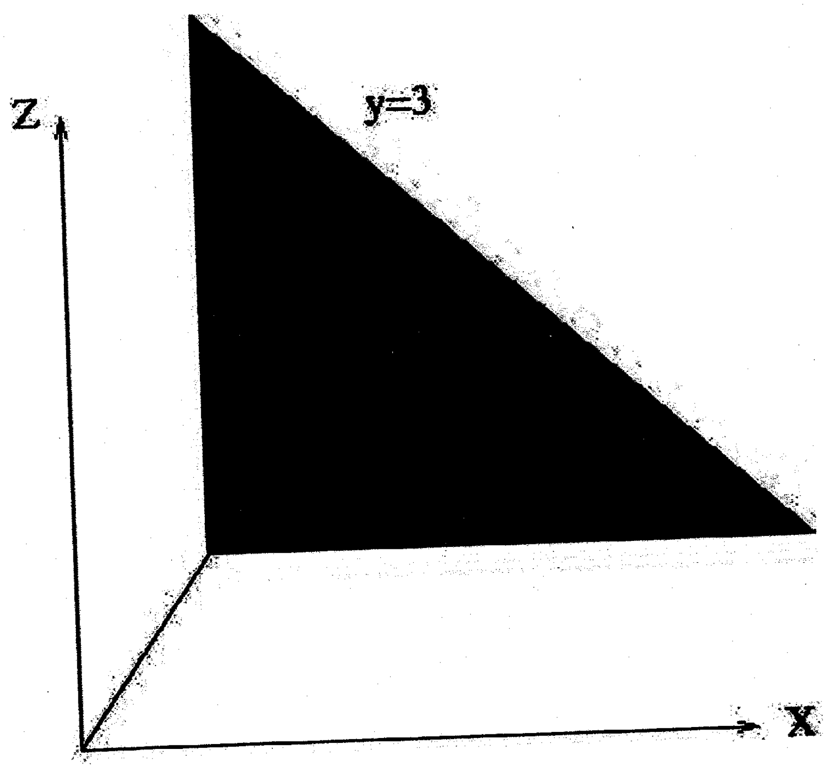
by folling rectangular path x-direction, y-direction

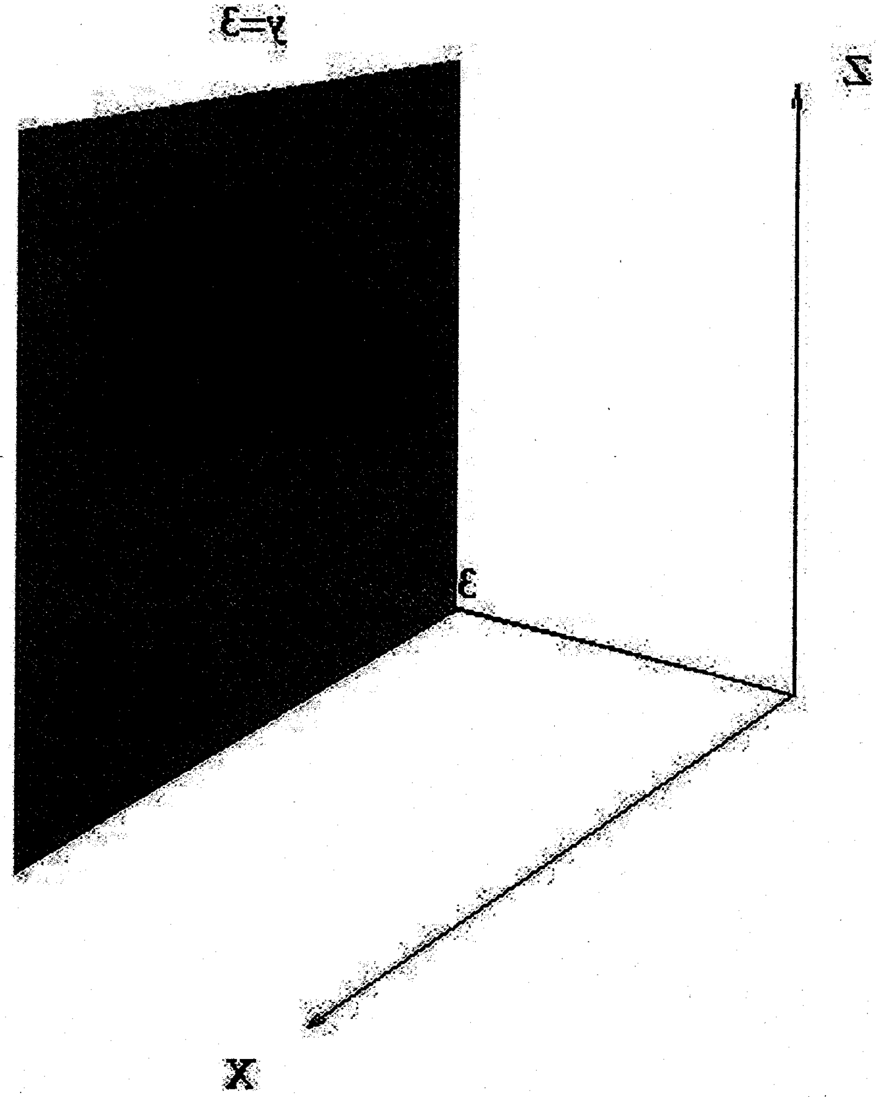
and z-direction

An equation in x, y, z represents

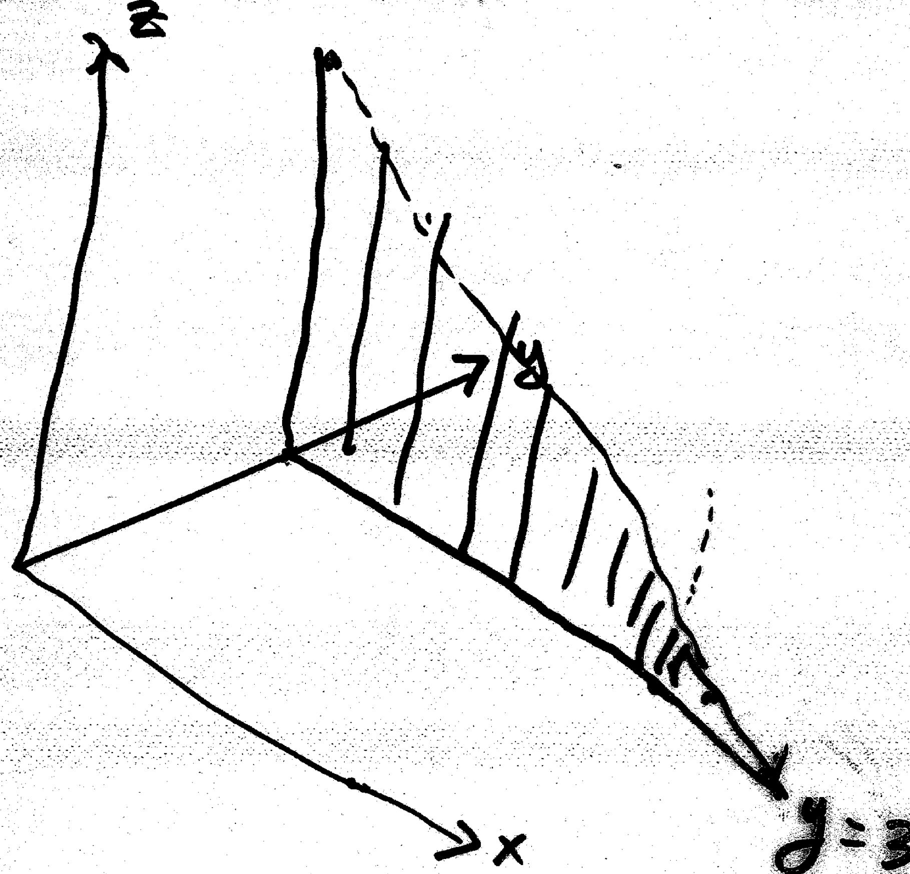
a <u>surface</u>.

<u>Ex</u>: $y = 3$ is a plane

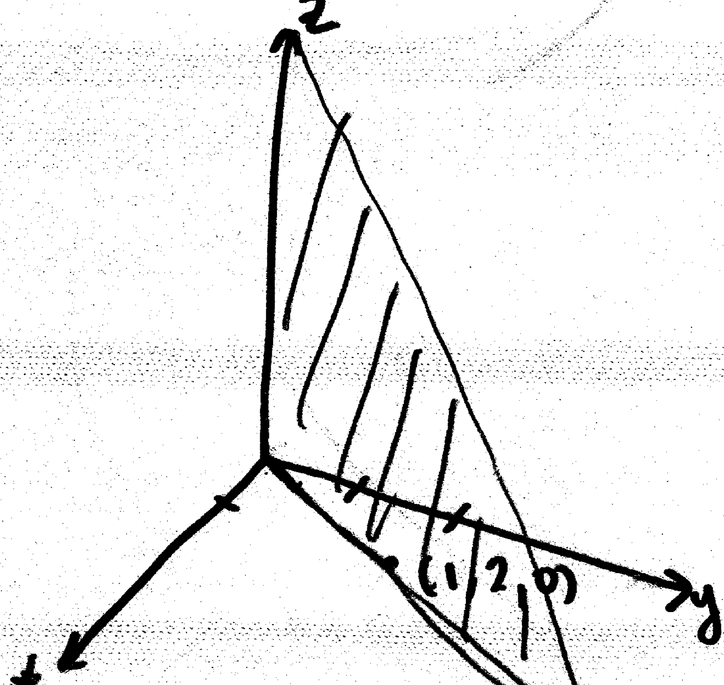

Z
y=3
X

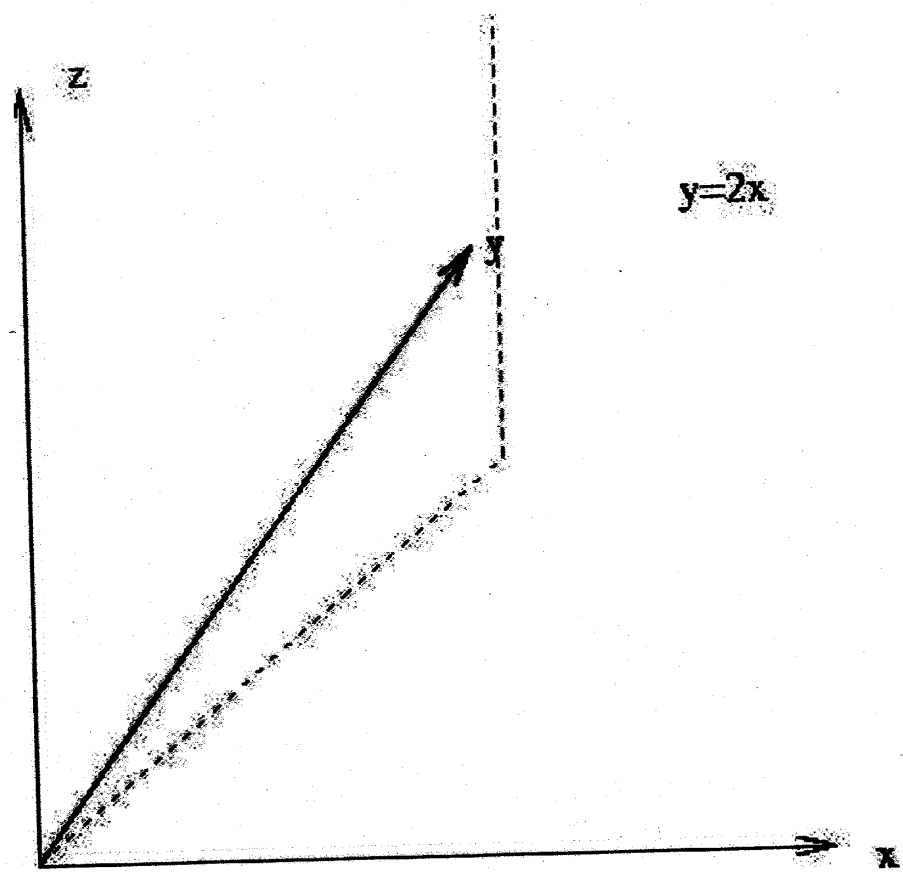

y=3
z
3
x

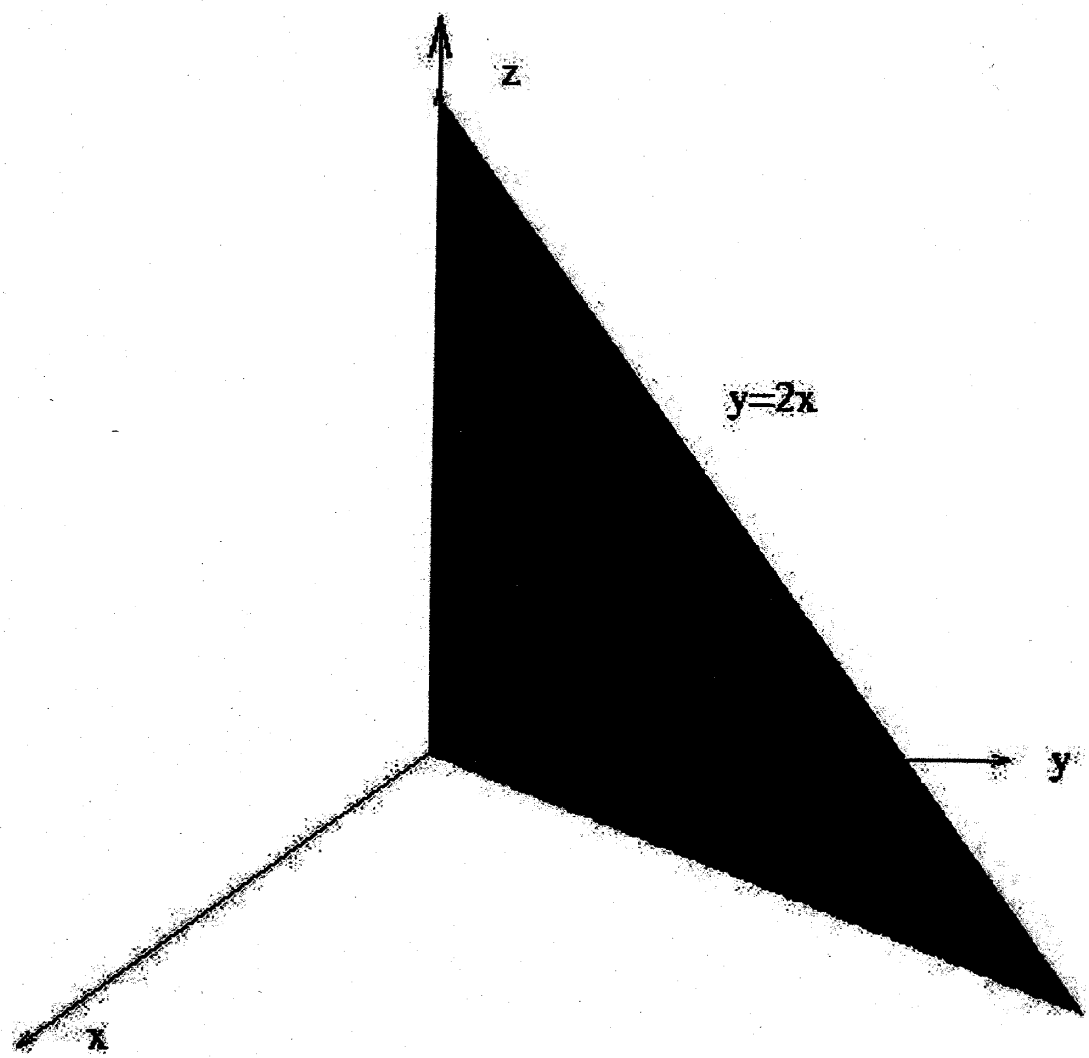
3



The equation $y = 2x$ is also a plane:




z
y=2x
y
x


z
y=2x
y
x

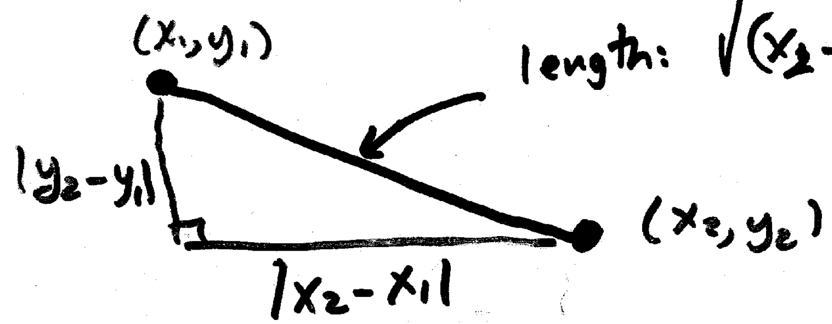
4

Distance:

The distance between two points in 3-space:

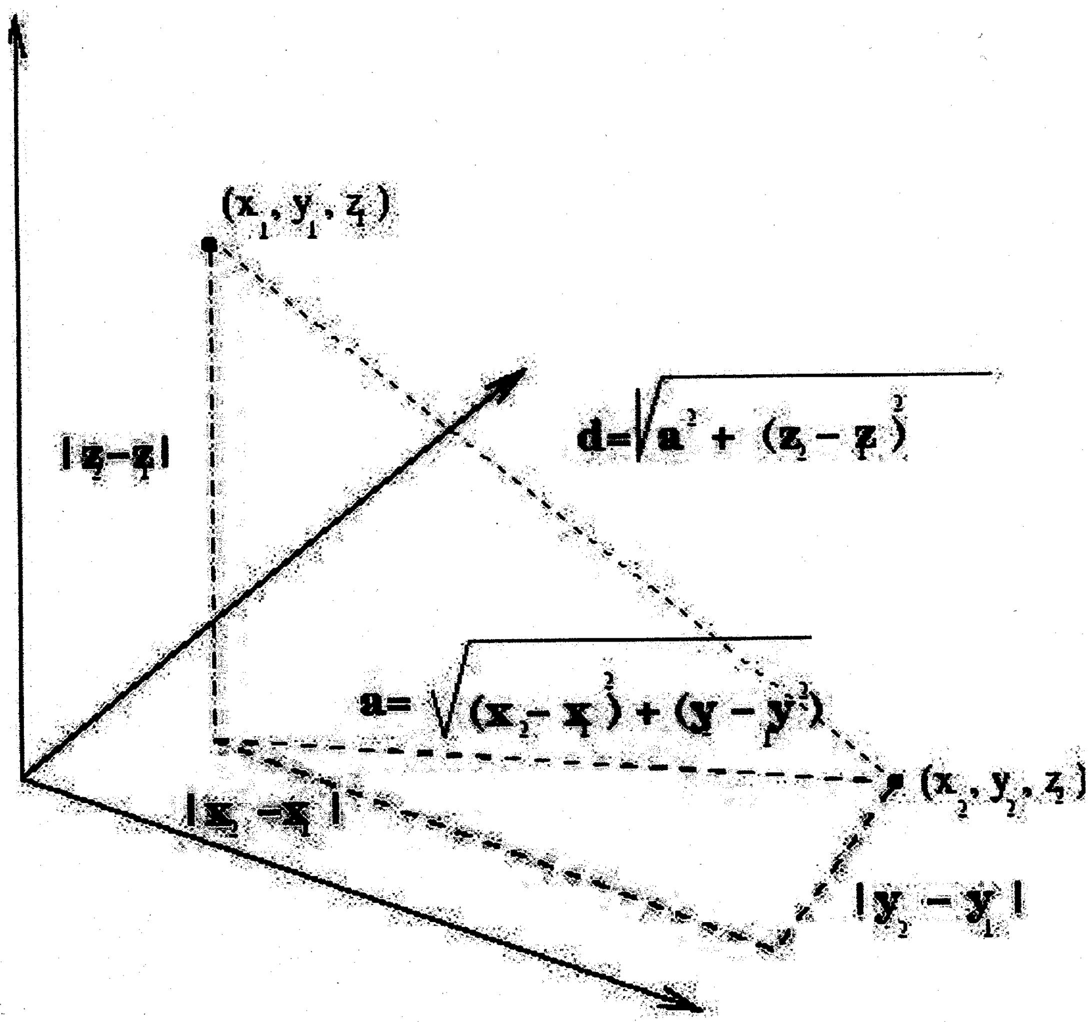
Recall: in $\mathbb{R}^2$: Distance between (x_1, y_1), (x_2, y_2).

length: $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$



In $\mathbb{R}^3$ the distance between two points $P_1(x_1, y_1, z_1)$, and $P_2(x_2, y_2, z_2)$ is:

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$


(x_1, y_1, z_1)
$|z_2 - z_1|$
$d = \sqrt{a^2 + (z_2 - z_1)^2}$
$a = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
(x_2, y_2, z_2)
$|x_2 - x_1|$
$|y_2 - y_1|$

Ex: $P_1(1, 2, -1)$, $P_2(3, -1, 4)$

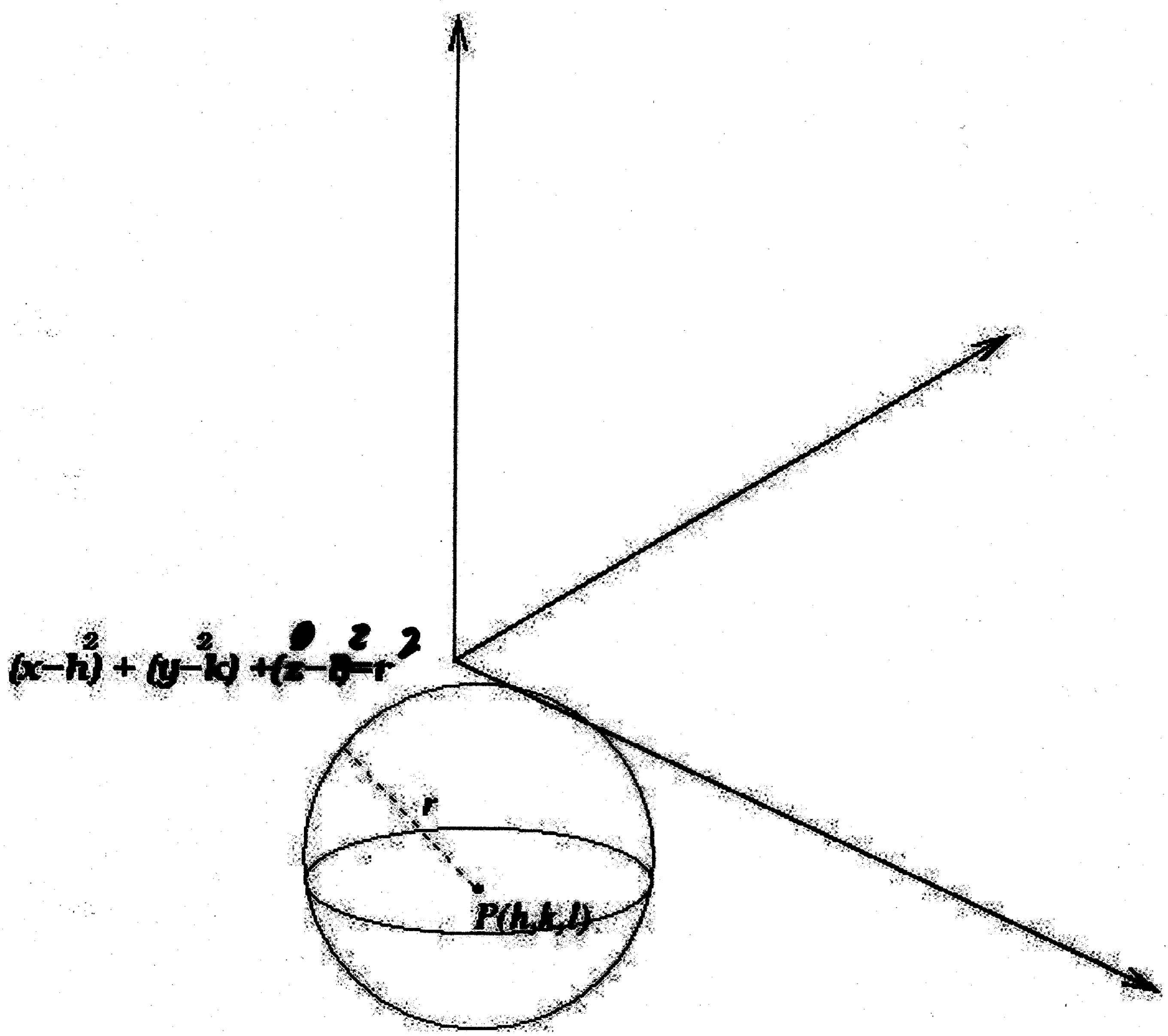
Distance:

$$\sqrt{(3-1)^2 + (-1-2)^2 + (4-(-1))^2}$$

$$= \sqrt{4+9+25} = \sqrt{38}$$

Just as a circle in $\mathbb{R}^2$ is the set of all points a fixed distance from a fixed point, a sphere in $\mathbb{R}^3$ is the set of all points in $\mathbb{R}^3$ a fixed distance (radius) from a fixed point (center). If r is the radius and the center is (h, k, ℓ), the sphere is all points with $(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2$

5a

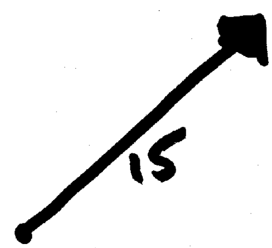


VECTORS

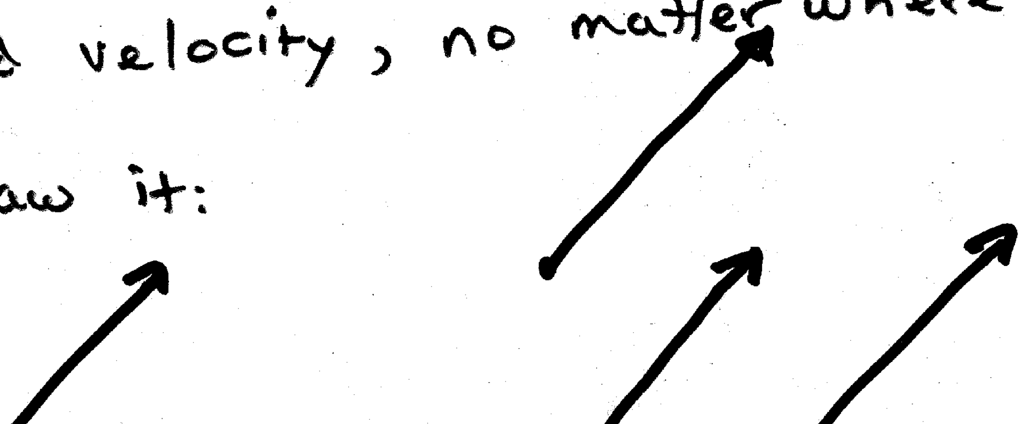
A vector represents two things

(i) size (or quantity) and
(ii) direction

Ex: Wind velocity; If the wind is blowing 15 mph to th Northeast we can represent the win by a vector, or arrow:



This arrow represents the san wind velocity, no matter where we draw it:

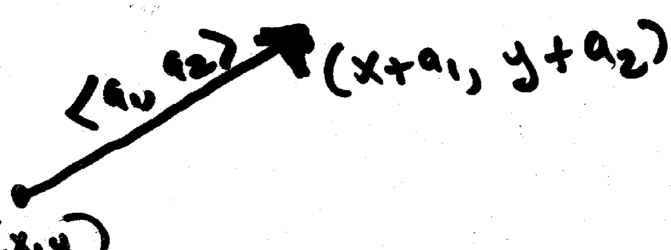


<u>Definition:</u> A <u>two dimensional</u> vector is an ordered pair $\underline{a} = \langle a_1, a_2$ of two real numbers.

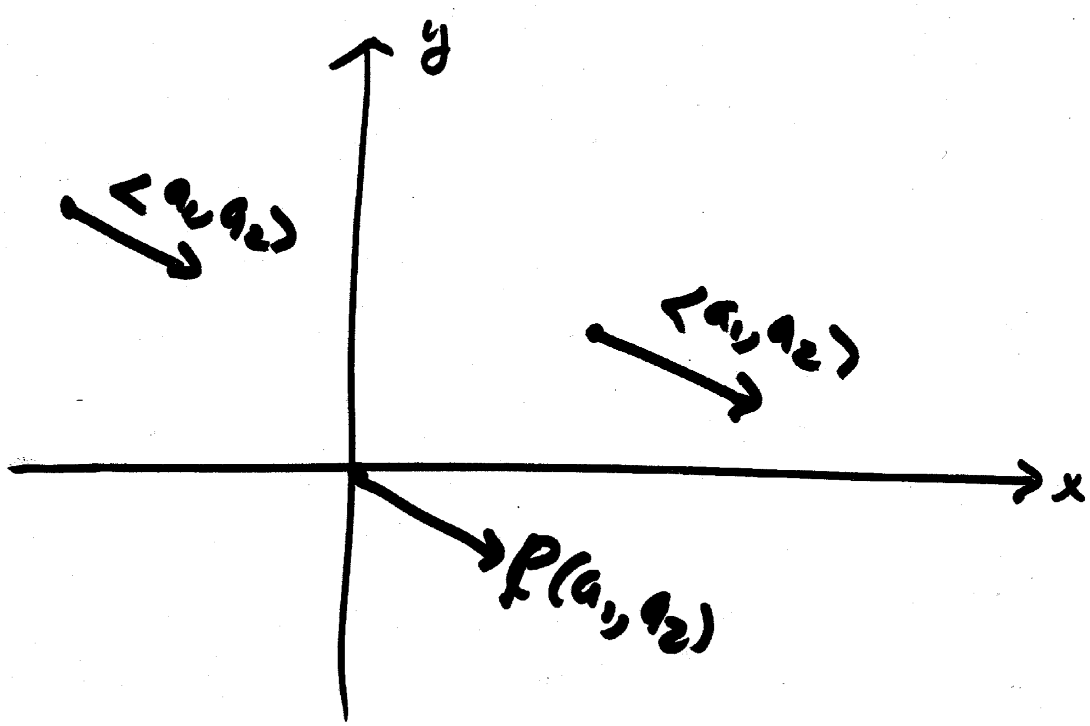
A <u>3-dimensional vector</u> is an ordered triple: $\underline{a} = \langle a_1, a_2, a_3 \rangle$

The numbers a_1, a_2, a_3 are called the <u>components</u> of $\underline{a}$.

We represent a vector $\underline{a} = \langle a_1, a_2 \rangle$ by a directed line segment $\overrightarrow{AB}$ from point $A(x, y)$ to $B(x+a_1, y+a_2$

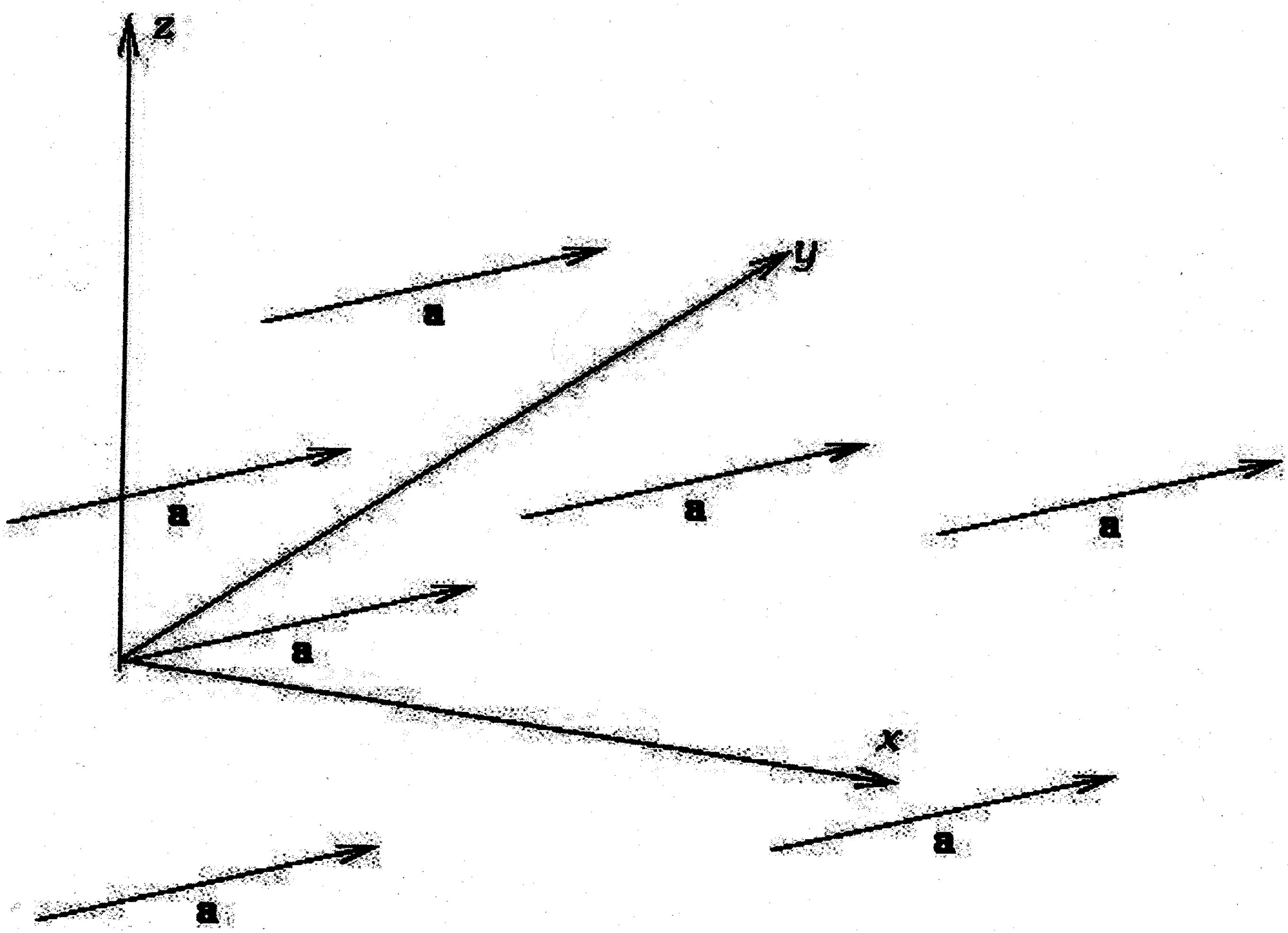


(8) The line segment $\overrightarrow{OP}$ from the origin to $P(a_1, a_2)$ is the <u>position</u> <u>vector</u> of $P(a_1, a_2)$

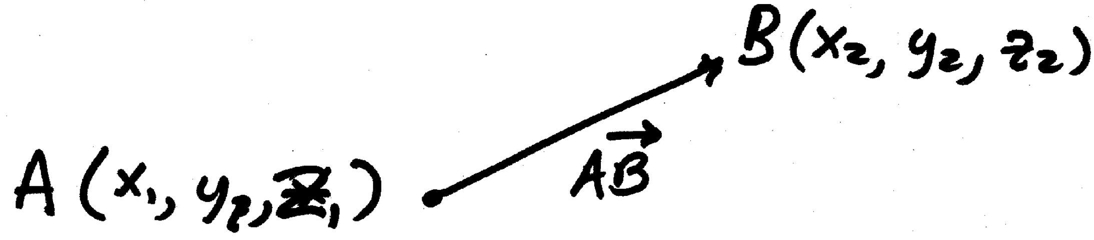


Similarly in 3 dimensions $\overrightarrow{OP}$ with $P = P(a_1, a_2, a_3)$ is the position vector $\langle a_1, a_2, a_3 \rangle$ of $P(a_1, a_2, a_3)$.

8a



9| If we know the initial and terminal points of a representation for a vector, then we can find the vector:

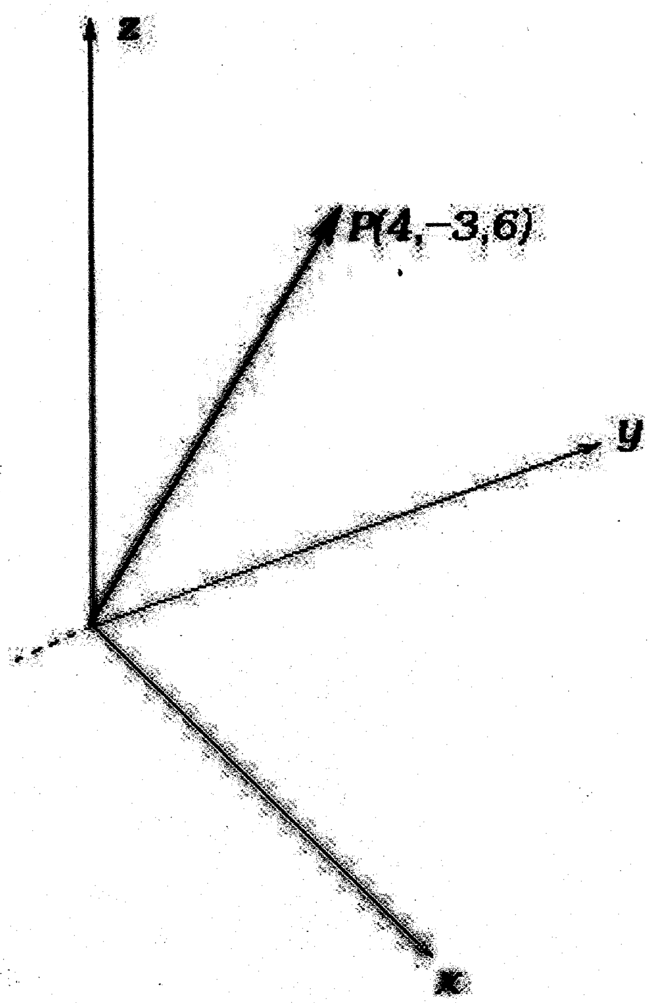


$\vec{AB} = \langle a_1, a_2, a_3 \rangle$, then

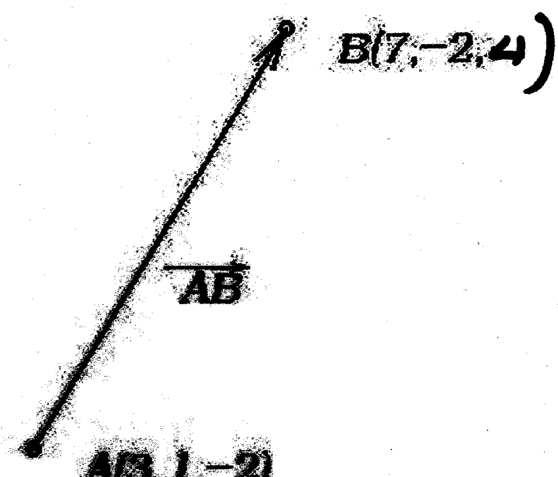
$$\begin{cases} x_1 + a_1 = x_2 \\ y_1 + a_2 = y_2 \\ z_1 + a_3 = z_3 \end{cases} \Rightarrow \quad \begin{aligned} a_1 &= x_2 - x_1 \\ a_2 &= y_2 - y_1 \\ a_3 &= z_2 - z_1 \end{aligned}$$

<u>**Example:**</u> The vector represented by the line from $A(3, 1, -2)$ to $B(7, -2, 4)$ is

$$\bar{a} = \langle 7-3, -2-1, 4-(-2) \rangle$$
$$= \langle 4, -3, 6 \rangle$$



z
P(4, −3, 6)
y
x



B(7, −2, 4)
AB
A(3, 1, −2)

Vectors have length defined by distance: i.e.

If $\underline{a} = \langle a_1, a_2 \rangle$

$|\underline{a}| = \sqrt{a_1^2 + a_2^2}$

$\underline{a} = \langle a_1, a_2, a_3 \rangle$

$|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

vector addition:

If $\underline{a} = \langle a_1, a_2 \rangle$; $\underline{b} = \langle b_1, b_2 \rangle$,

then $\underline{a} + \underline{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$.

Similarly $\underline{a} = \langle a_1, a_2, a_3 \rangle$

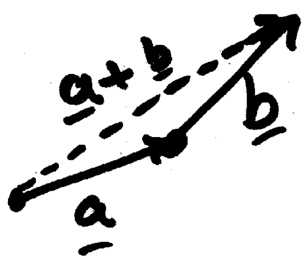
$\underline{b} = \langle b_1, b_2, b_3 \rangle$

then $\underline{a} + \underline{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

11

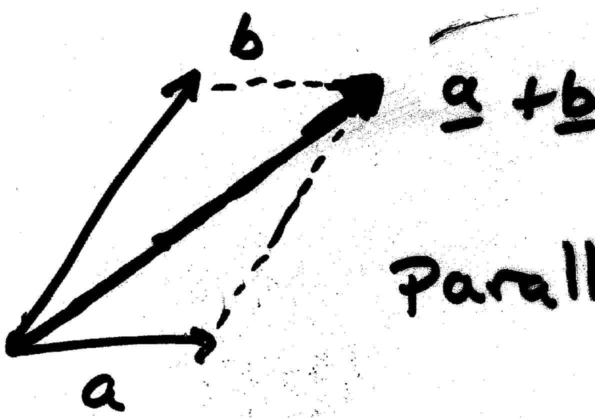
Representation

Place one initial point at the terminal point of the other:



Triangle law

Or: draw both at same initial point:



Parallelogram law

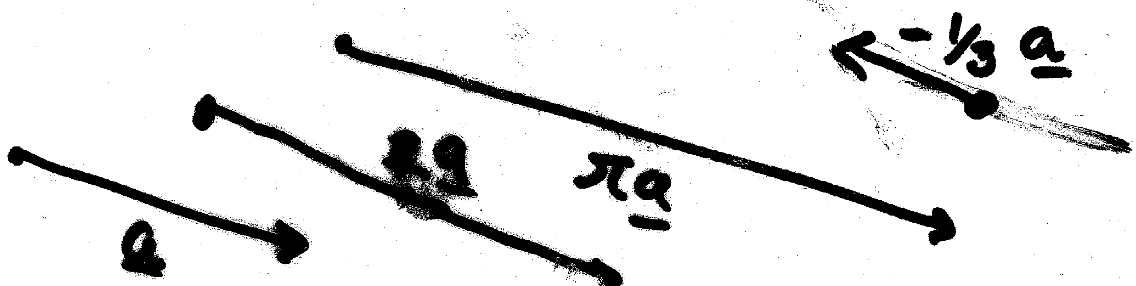
Scalar multiplication.

if c is a real numbe and $\underline{a}$ is a vector, we define a new vector ; $c\underline{a}$ by multiplying each component of $\underline{a}$ by c.

$$\underline{a} = \langle a_1, a_2, a_3 \rangle$$

$$c\,\underline{a} = \langle ca_1, ca_2, ca_3 \rangle$$

Two vectors $\underline{a}$ and $\underline{b}$ are called <u>parallel</u> if $\underline{a} = c\underline{b}$ for some scalar c.

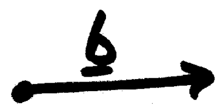


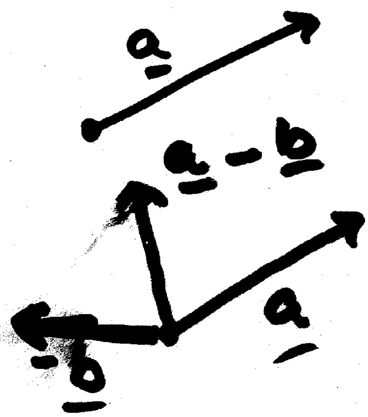
B

If $\underline{a}$ and $\underline{b}$ are vectors, then

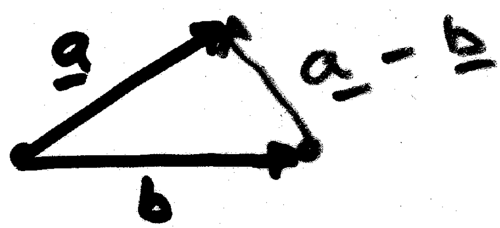
$$\underline{a} - \underline{b} = a + (-b), \text{ where}$$

$$-b = -1b \quad (\text{Scalar times vector}).$$





or



The zero vector $\underline{0} = \langle 0, 0 \rangle$ (2-dim)

$\underline{0} = \langle 0, 0, 0 \rangle$

Properties:

1. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

2. $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$

3. $\underline{a} + \underline{0} = \underline{a}$

4. $\underline{a} + (-\underline{a}) = \underline{0}$

5. $c(\underline{a} + \underline{b}) = c\underline{a} + c\underline{b}$

6. $(c+d)\,\underline{a} = c\underline{a} + c\underline{b}$

7. $(cd)\,\underline{a} = c(d\underline{a})$

8. $1 \cdot \underline{a} = \underline{a}$

These are <u>easily</u> <u>checked</u>, but you should be careful to know what each symbol <u>is</u>.

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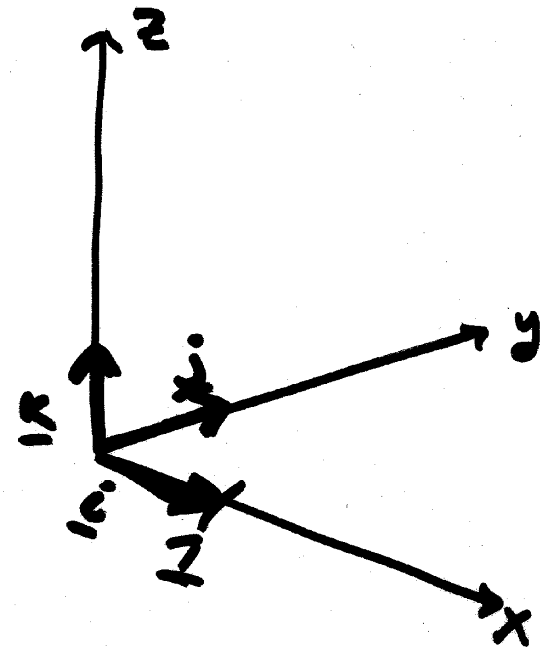
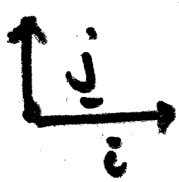
Standard basis vectors

$\underline{i} = \langle 1, 0, 0 \rangle$; $\underline{j} = \langle 0, 1, 0 \rangle$

$\underline{k} = \langle 0, 0, 1 \rangle$



(in 2 dimensions, $\underline{i} = \langle 1, 0 \rangle$
$\underline{j} = \langle 0, 1 \rangle$)



If $\underline{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$$

Ex: $\langle 3, -1, 5 \rangle =$

$$3\underline{i} - \underline{j} + 5\underline{k}$$

Ex: $\underline{a} = 2\underline{i} + 4\underline{j} - 3\underline{k}$

$\underline{b} = 4\underline{i} - 2\underline{j} + 5\underline{k}$

Find $2\underline{a} - 4\underline{b}$

$$2\underline{a} - 4\underline{b} = 2(2i + 4j - 3\underline{k}) - 4(4\underline{i} - 2\underline{j} + 5\underline{k})$$

$$= (4\underline{i} + 8\underline{j} - 6\underline{k}) - (16\underline{i} - 8\underline{j} + 20\underline{k})$$

$$= -12\underline{i} + 16\underline{j} - 26\underline{k}$$

A <u>unit vector</u> is a vector of length 1.

For any vector $\underline{a}$ there is a <u>unit vector</u> <u>in the direction of</u> $\underline{a}$.

Namely, $\underline{u} = \frac{1}{|\underline{a}|} \cdot \underline{a}$

Ex: Unit vector in the direction of

$\underline{a} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

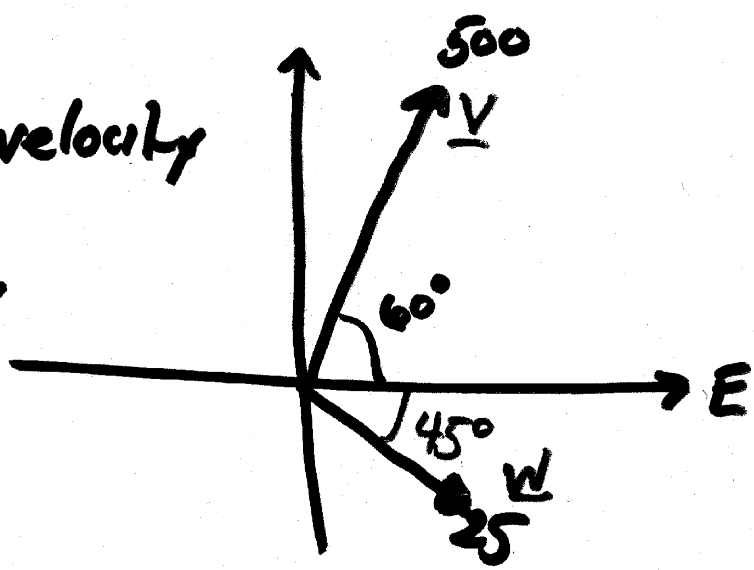
$$|\underline{a}| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$$

$$\underline{u} = \frac{1}{|\underline{a}|}\underline{a} = \frac{1}{\sqrt{29}}(2\underline{i} - 3\underline{j} + 4\underline{k})$$

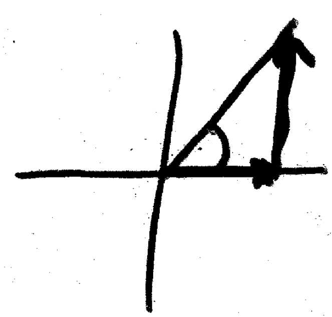
$$= \frac{2}{\sqrt{29}}\underline{i} - \frac{3}{\sqrt{29}}\underline{j} + \frac{4}{\sqrt{29}}\underline{k}.$$

Ex: A plane flies with airspeed 500 m.p.h in the direction 60°N of East. If there is a wind blowing, 25 mph in the direction 45° S of east, find the ground ~~velocity~~ velocity of the plane.

Let $\underline{v}$ be the air ~~speed~~ velocity and $\underline{w}$ the wind.



Note: $\underline{v} = a\underline{i} + b\underline{j}$;



$\underline{a} = 500 \cdot \cos 60° = 250$

$\underline{b} = 500 \cdot \sin 60° = 250\sqrt{3}$

$W = 25\sqrt{2}\, i - 25\sqrt{2}\, j$

Ground velocity is the

$$\underline{V}_g = \left(250 + \frac{25}{2}\sqrt{2}\right)\underline{i} + \left(\frac{250\sqrt{3} - 25\sqrt{2}}{2}\right)\underline{j}$$

Dot Product

There is, in general, no reasonable way to multiply two vectors to get a new vector. HOWEVER: There is a way to assign a number (i.e. scalar) as the product of two vectors. This is the dot product

2-d: $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

3-d $\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

Ex:

$\langle 2,-1\rangle \cdot \langle 3,5\rangle = 2\cdot 3 + (-1)\cdot 5$

$= 1$

$(4\underline{i} - \underline{j} + \underline{k}) \cdot (5\underline{i} + 3\underline{j}) =$

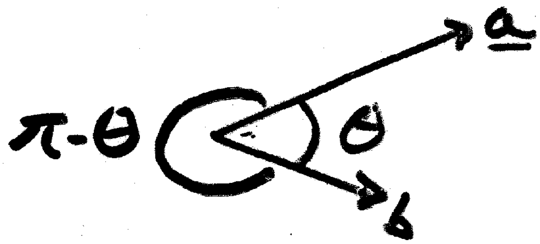
$4\cdot 5 + (-1)(3) + 1\cdot 0 = 17$

Properties:

1. $\underline{a}\cdot\underline{a} = |\underline{a}|^2$
2. $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$
3. $\underline{a}\cdot(\underline{b}+\underline{c}) = (\underline{a}\cdot\underline{b}) + (\underline{a}\cdot\underline{c})$
4. $\underline{0}\cdot\underline{a} = \underline{0}$
5. $c\underline{a}\cdot b = c(\underline{a}\cdot\underline{b}) = \underline{a}\cdot(c\underline{b})$.

Why should we care? i.e. So what if we can assign a value to two vectors, does it have a meaning?

Def: The angle, θ, between $\underline{a}$ and $\underline{b}$ is defined to be the angle given when both $\underline{a}$ and $\underline{b}$ are drawn at the origin, with $0 \le \theta \le \pi$.



Theorem: The angle θ between $\underline{a}$ and $\underline{b}$ is given by

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| \cdot |\underline{b}|}$$

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Ex: Find the angle between

$\underline{a} = \langle 1, -1, 3 \rangle$ and $\underline{b} = \langle 5, 1, -1 \rangle$

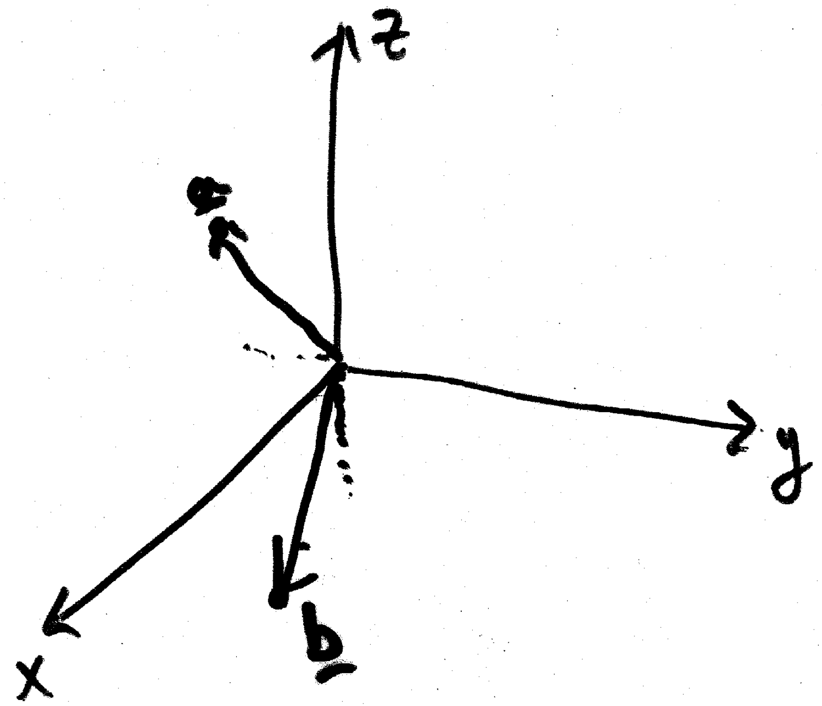
Soln: $|\underline{a}|^2 = 1^2 + (-1)^2 + 3^2 = 11$

$|\underline{b}|^2 = 5^2 + 1^2 + (-1)^2 = 27$

$\underline{a} \cdot \underline{b} = 1 \cdot 5 + (-1) \cdot (1) + 3(-1) = 1$

$$\cos\theta = \frac{a \cdot b}{|\underline{a}|\,|\underline{b}|} = \frac{1}{\sqrt{297}}$$

$$\theta = \arccos\left(1/\sqrt{297}\right) \approx 86.67^\circ$$



23 **Note:** $\underline{a}$ and $\underline{b}$ are orthogonal if and only if $\underline{a} \cdot \underline{b} = 0$

Ex: Find the scalar c so that $\underline{a} = 2\underline{i} + 3\underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + c\underline{k}$ are orthogonal.

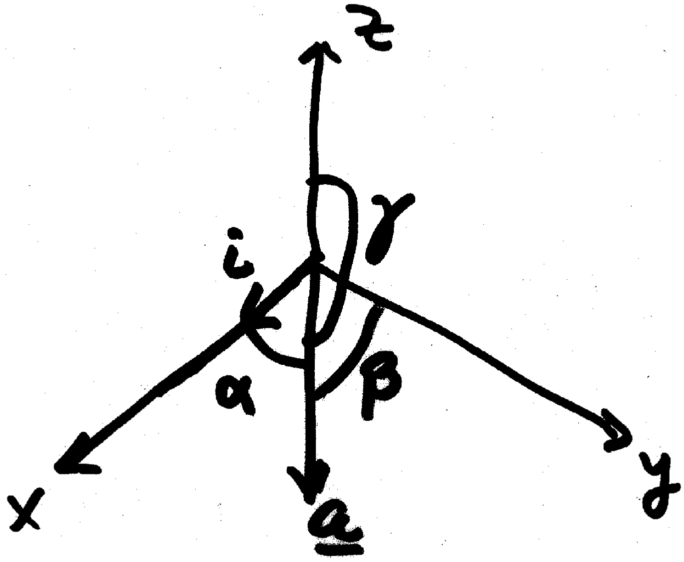
Since $\underline{a} \cdot \underline{b} = 2 - 3 - c = -1 - c$, and we need $\underline{a} \cdot \underline{b} = 0$, we must have $c = -1$.

$\underline{a} = \langle 2, 3, -1 \rangle \quad \underline{b} = \langle 1, -1, -1 \rangle$.

Direction angles and cosines

The direction angles of a non-zero vector (3-dimensional) are the angles α, β, and γ that $\underline{a}$ makes with the positive x, y, and z axes.

Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the <u>direction cosines</u> of $\underline{a}$.



If $\underline{a} = \langle a_1, a_2, a_3 \rangle$

<u>Note:</u> $\cos\alpha = \dfrac{\underline{a}\cdot\underline{i}}{|\underline{a}||\underline{i}|} = \dfrac{a_1}{|\underline{a}|}$

Similarly,

$$\cos\beta = \frac{a_2}{|\underline{a}|}$$

$$\cos\gamma = \frac{a_3}{|\underline{a}|}$$

So
$$a_1 = |\underline{a}| \cos\alpha$$
$$a_2 = |\underline{a}| \cos\beta$$
$$a_3 = |\underline{a}| \cos\gamma,$$

and thus

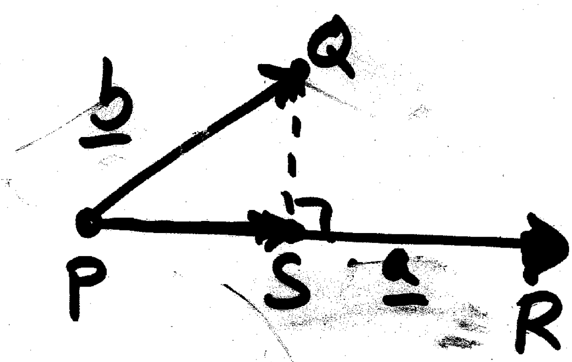
$$\underline{a} = |\underline{a}| \langle \underline{\cos\alpha, \cos\beta, \cos\gamma} \rangle$$

<u>Note:</u> $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

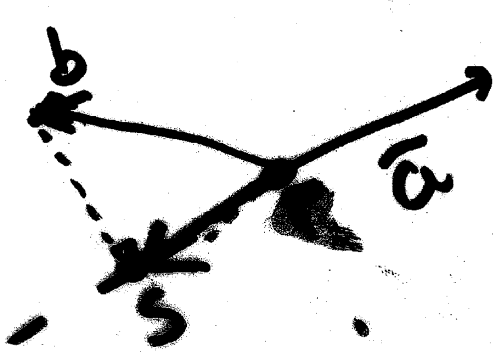
So $\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ is the unit vector in the direction of $\underline{a}$.

Projection of one vector onto another

If we represent two vectors $\underline{a}$ and $\underline{b}$ with the same initial point, P, and S is the foot of the perpendicular from $\underline{a}$ to $\underline{b}$, then $\overrightarrow{PS}$ is called the vector projection of $\underline{b}$ onto $\underline{a}$.



$\text{Proj}_{\underline{a}}\,\underline{b}$



27 The scalar projection of $\underline{b}$ onto $\underline{a}$ is the number c so that

$$\text{proj}_{\underline{a}}\,\underline{b} = c \cdot \frac{\underline{a}}{|\underline{a}|}.$$

We call c the component of $\underline{b}$ along $\underline{a}$, $\text{comp}_{\underline{a}}\,\underline{b}$. Note

$$\text{comp}_{\underline{a}}\,\underline{b} = |\underline{b}|\cos\theta,$$

with θ as before.

Thus

$$\text{comp}_{\underline{a}}\,\underline{b} = |b|\cos\theta =$$

$$\frac{\underline{a}\cdot\underline{b}}{|a|}$$

So

$$\text{proj}_{\underline{a}}\,\underline{b} = \left(\frac{\underline{a}\cdot\underline{b}}{|\underline{a}|}\right)\frac{\underline{a}}{|\underline{a}|} = \left(\frac{\underline{a}\cdot\underline{b}}{|a|^2}\right)\underline{a}$$

Ex: $\underline{a} = \langle 3, 2, -4 \rangle$

$\underline{b} = \langle 1, 0, 2 \rangle$

$$\text{Proj}_{\underline{a}}\, \underline{b} = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|^2}\, \underline{a} =$$

$$\frac{(3-8)}{29} \cdot \langle 3, 2, -4 \rangle =$$

$$\frac{-5}{29} \langle 3, 2, -4 \rangle = \left\langle \frac{-15}{29}, \frac{-10}{29}, \frac{20}{29} \right\rangle$$

This is used to decompose forces

(see book)

Cross Product

For 3-dimensional vectors, AND ONLY 3-dimensional vectors, there is a vector product, called the <u>cross product</u>.

If $\underline{a} = \langle a_1, a_2, a_3 \rangle$, $\underline{b} = \langle b_1, b_2, b_3 \rangle$

then

$\underline{a} \times \underline{b} =$

$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1$

Can be remembered as

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend left] \end{tikzcd}$$

Another way to remember is to use <u>determinants</u>

For a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

the determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A 3x3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Ex:

I. $\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 3\cdot 4 - 2\cdot 1 = 12-2$
$= 10$

II. $\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ -1 & 4 & -2 \end{vmatrix} =$

$$1\cdot\begin{vmatrix} 1 & 4 \\ 4 & -2 \end{vmatrix} - 0\cdot\begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix}$$

$= (-2-16) - 0 + 3(8+1)$

$= 9$

Determinants have important properties (e.g. Cramer's rule), you will learn more about determinants in MA 261, 265, 266, 303 etc.

One useful property is:

If $\underline{a} = \langle a_1, a_2, a_3 \rangle$, $\underline{b} = \langle b_1, b_2, b_3$

then $\underline{a} \times \underline{b} =$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex: $\underline{a} = \langle 2, -7, 3 \rangle$, $\underline{b} = \langle 1, 4, 1 \rangle$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -7 & 3 \\ 1 & 4 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} -7 & 3 \\ 4 & 1 \end{vmatrix} \underline{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \underline{j} + \begin{vmatrix} 2 & -7 \\ 1 & 4 \end{vmatrix} \underline{k}$$

$8 - -7$

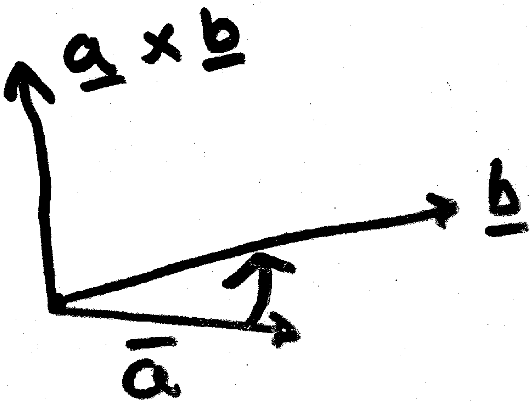
$$= -19\underline{i} + \underline{j} + \underline{15}\,\underline{k}$$

Why do we care??

The cross product has some nice properties

THM: $\underline{a} \times \underline{b}$ is orthogonal to both $\underline{a}$ and $\underline{b}$.

Moreover $\underline{a} \times \underline{b}$ has direction dictated by the right hand rule:



Ex. Find a unit vector perpendicular to both $\langle 1, 3, 2 \rangle$ and $\langle -1, 4, 2 \rangle$

Let

$$\underline{v} = \langle 1, 3, 2 \rangle \times \langle -1, 4, 2 \rangle =$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 2 \\ -1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} \underline{i}$$

$$- \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \underline{j} + \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} \underline{k}$$

$$= -2\underline{i} - 4\underline{j} + 7\underline{k} .$$

Then our unit vector is

$$\underline{u} = \frac{\underline{v}}{|\underline{v}|} = \frac{-2}{\sqrt{69}}\underline{i} - \frac{4}{\sqrt{69}}\underline{j} + \frac{7}{\sqrt{69}}\underline{k}.$$

<u>Good to check</u>: $\underline{a} \times \underline{b}$ is perp. to both $\underline{a}$ and $\underline{b}$

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$\langle -2, -4, 7 \rangle \cdot \langle 1, 3, 2 \rangle = -2 - 12 + 14$
$= 0$

$\langle -2, -4, 7 \rangle \cdot \langle -1, 4, 2 \rangle = 2 - 16 + 14 = 0.$

Magnitude:

<u>Thm</u> $|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}| \cdot \sin\theta$

where θ is the angle between $\underline{a}$ and $\underline{b}$.

<u>Corollary</u> $\underline{a}$ is parallel to $\underline{b}$ $\Leftrightarrow$ $\underline{a} \times \underline{b} = 0$.

<u>Note:</u> $|\underline{a} \times \underline{b}|$ is the area of the parallelogram formed by $\underline{a}$ and $\underline{b}$.

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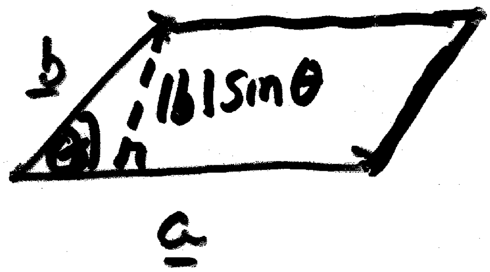
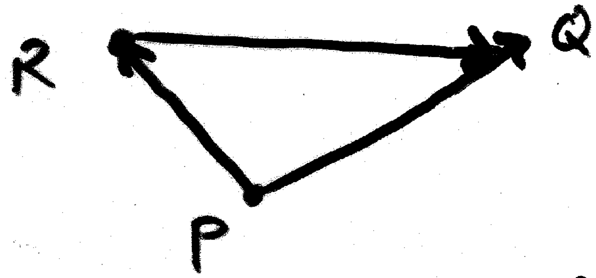


Ex: Find the area of the triangle with vertices $P(1,2,1)$, $Q(2,-1,3)$ and $R(3,1,1)$



$\vec{PQ} = \langle 1, -3, 2 \rangle$

$\vec{PR} = \langle 2, -1, 0 \rangle$

The area we want is half that of the parallelogram formed by $\vec{PQ}$ and $\vec{PR}$. Its area is $|\vec{PQ} \times \vec{PR}|$, Then take half.

37. **Note:**

① $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$

② $(\underline{a} \times \underline{b}) \times \underline{c} \neq \underline{a} \times (\underline{b} \times \underline{c})$.

So not all usual algebra rules hold. BE CAREFUL!!

EX: $(\underline{i} \times \underline{j}) \times \underline{j} = \underline{k} \times \underline{j} = -\underline{i}$

$i \times (\underline{j} \times \underline{j}) = \underline{i} \times \underline{0} = \underline{0}$.

Some other properties do hold:

(i) $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

(ii) $(\alpha \underline{a}) \times \underline{b} = \alpha (\underline{a} \times \underline{b})$

$= \underline{a} \times (\alpha \underline{b})$.

Where α is a scalar.

etc. See Pg. 807

The volume of the parallelopiped formed by $\underline{a}$, $\underline{b}$ and $\underline{c}$ is the magnitude of their scalar product; $|\underline{a} \cdot (\underline{b} \times \underline{c})|$

So $\underline{a}$, $\underline{b}$, $\underline{c}$ are <u>coplanar</u>

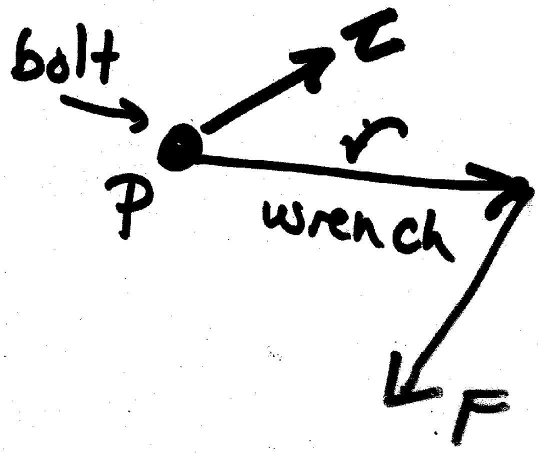
$$\Longleftrightarrow \quad \underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{0}$$

<u>Note</u> Property 5 on pg. 807

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$$

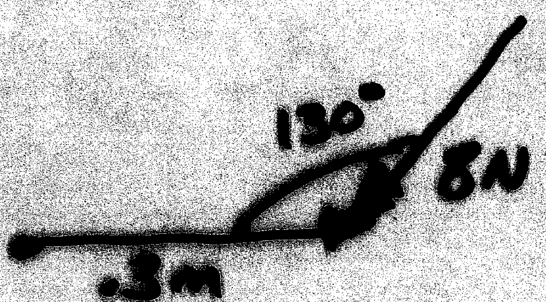
An example of an application of the cross product is <u>torque</u>

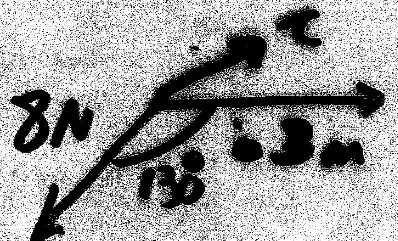
If a force F acts on a rigid body (e.g. a wrench) with position vector $\underline{r}$ the <u>torque</u> is $\tau = r \times F$



Example:

A wrench of lenght 30cm is used to tighten a bolt. If The hand forms an angle of 130° with the wrench, and applies a force of 8 Newtons, what is the torque?





$$|\tau| = |F||r| \sin 130^\circ$$
$$= 8(.3) \cdot \approx 1.83\ J$$