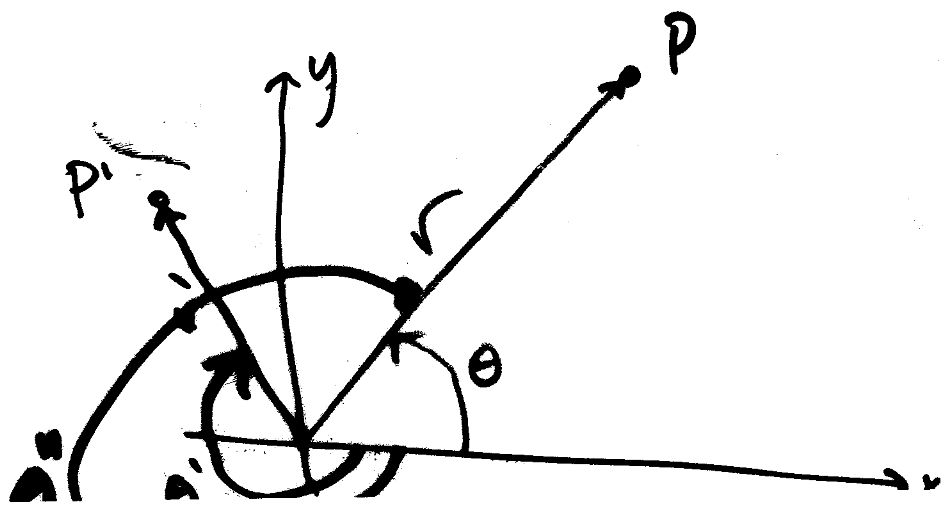
Polar Coordinates

Instead of referring to a point P by its (x,y)-coordinates, we can use the origin as a "pole" and the positive x-axis as a polar axis and refer to a point by a distance and an angle.

That is refer to a point by a distance and direction.

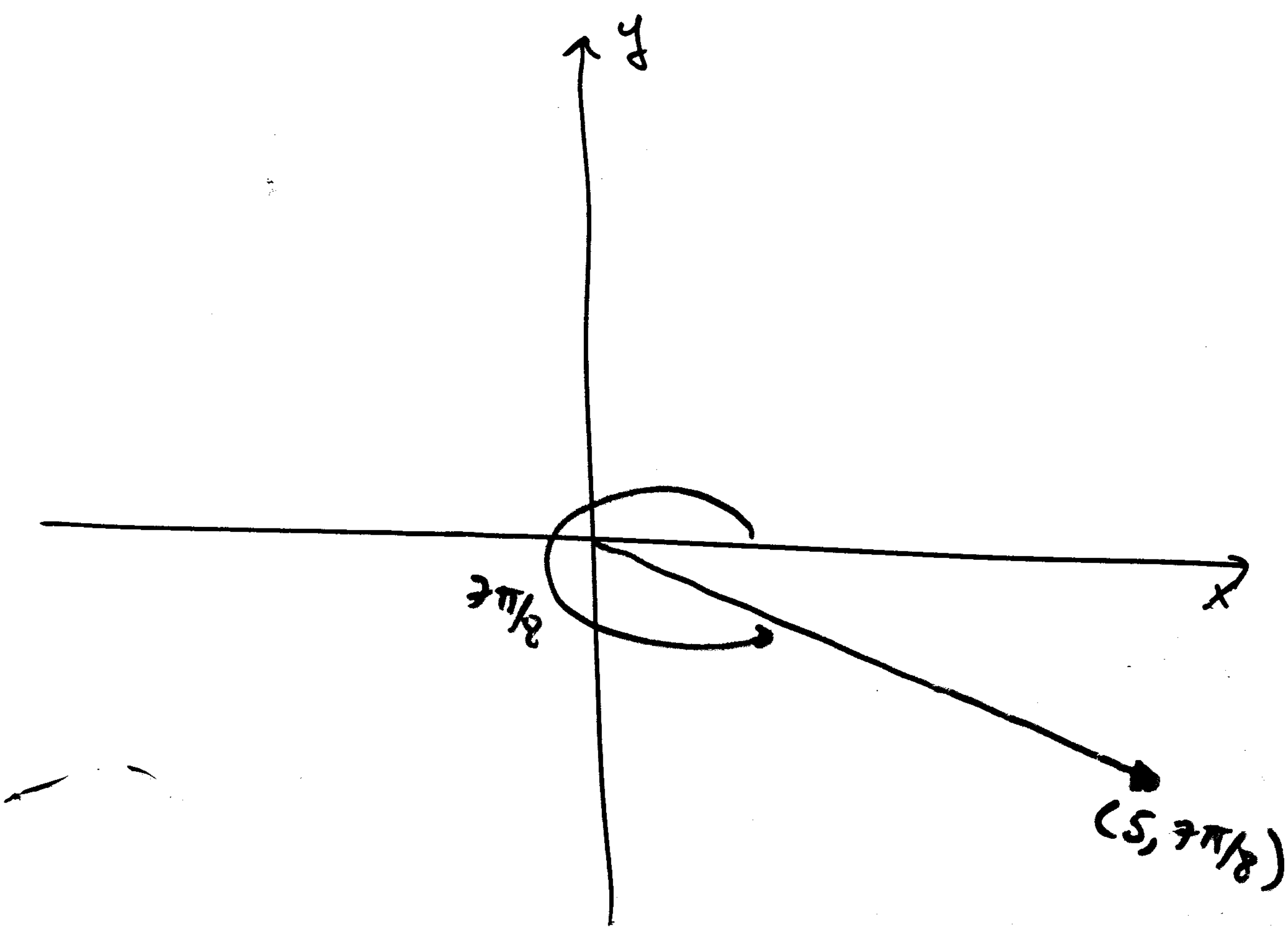


Example:

Find the points

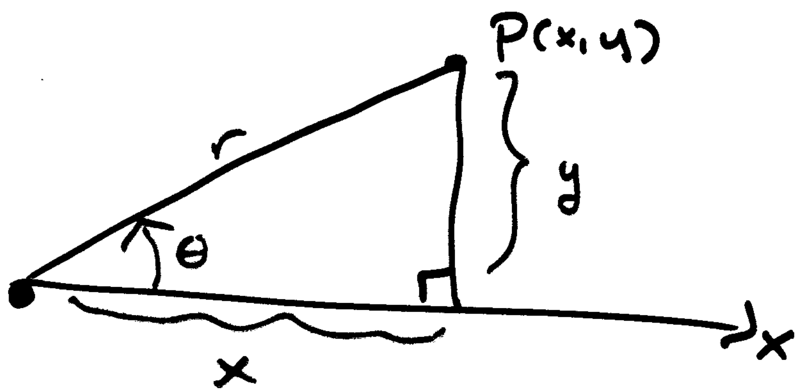
$(5, 7\pi/8)$, $(2/3, 5\pi/3)$,

$(-1, \pi/4)$, $(4, \pi/6)$, $(2, 8\pi/3)$



Converting between Coordinate Systems:

If P has coordinates (x, y) then its polar coordinates are



$\cos\theta =$ $\sin\theta =$

$x =$

$y =$

If we know $P(x,y)$, then

$r^2 = x^2+y^2$, $\tan\theta = y/x$

Convert $(2, 5\pi/3)$ to Cartesian coordinates

$x =$

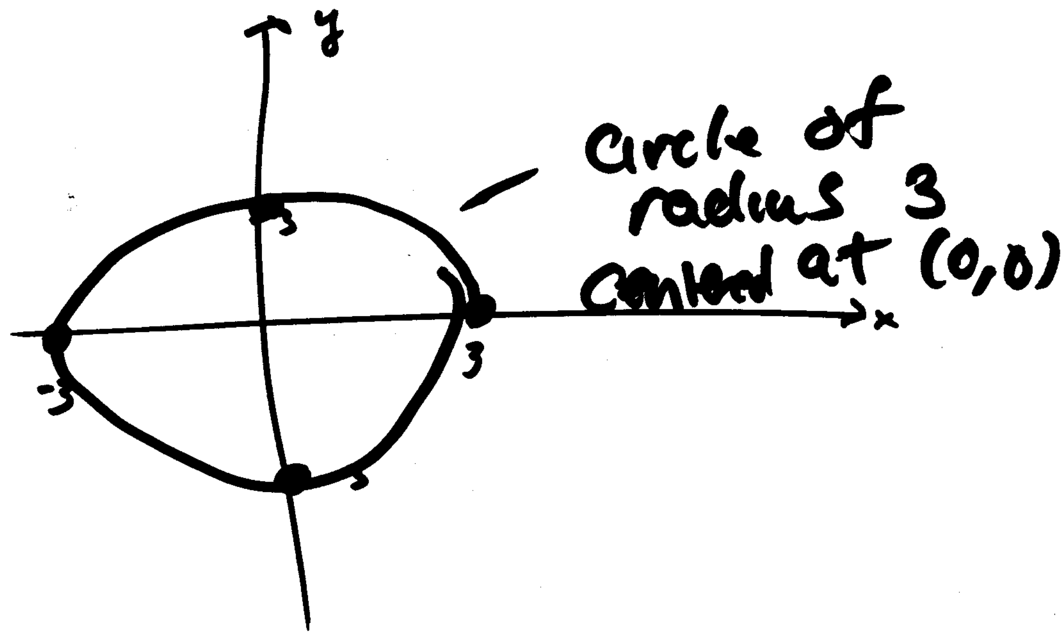
$y =$

Convert $(2,2)$ to polar coordinates

$r =$ $\theta =$

A <u>graph</u> of an equation $r = f(\theta)$ is the points (r, θ) satisfying the given equation:

<u>Ex</u>: $r = 3$;



Sometimes the equation may be:

$\theta = F(r)$, or $G(r, \theta) = 0$

<u>Ex</u>: $\theta = \pi/6$;

Example: $r = 4\sin\theta$

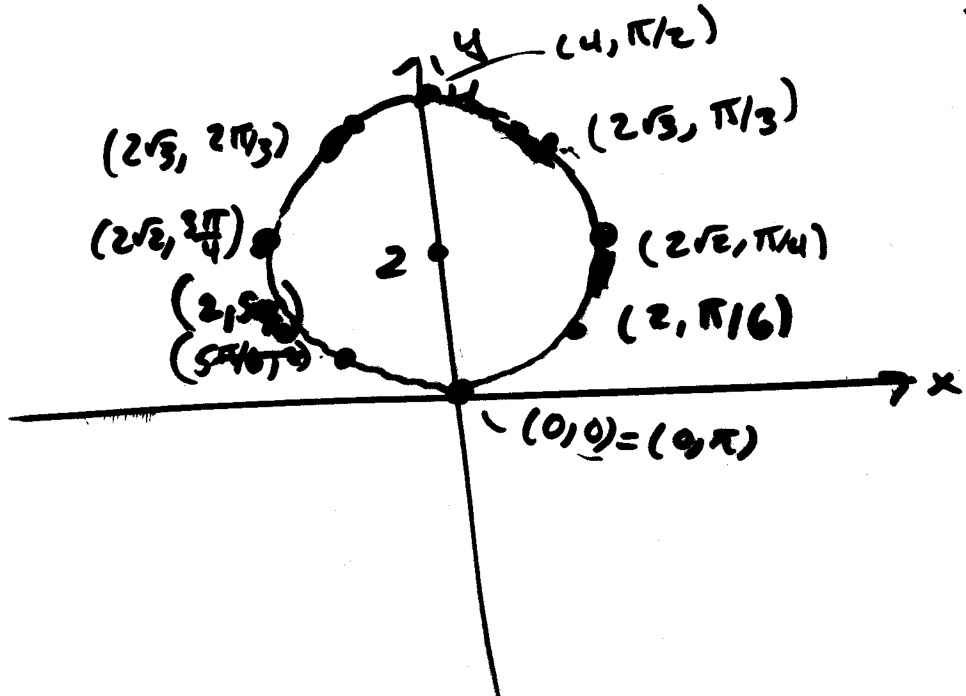
Note: Since $r(\theta+\pi) = 4\sin(\theta+\pi) = -4\sin\theta,$
$= -r(\theta)$

the point $(-r, \theta+\pi) = (r,\theta)$.

So we can find the graph by looking at

$0 \le \theta \le \pi$

θ	r
0	0
$\frac{\pi}{6}$	2
$\frac{\pi}{4}$	$2\sqrt{2}$
$\pi/3$	$2\sqrt{3}$
$\pi/2$	4
$2\pi/3$	$2\sqrt{3}$
$3\pi/4$	$2\sqrt{2}$
$5\pi/6$	2
π	0



Is it really a circle?

$r = 4\sin\theta$; $\quad y = r\sin\theta$

$\frac{r}{4} = \sin\theta = \frac{y}{r} \Longrightarrow$

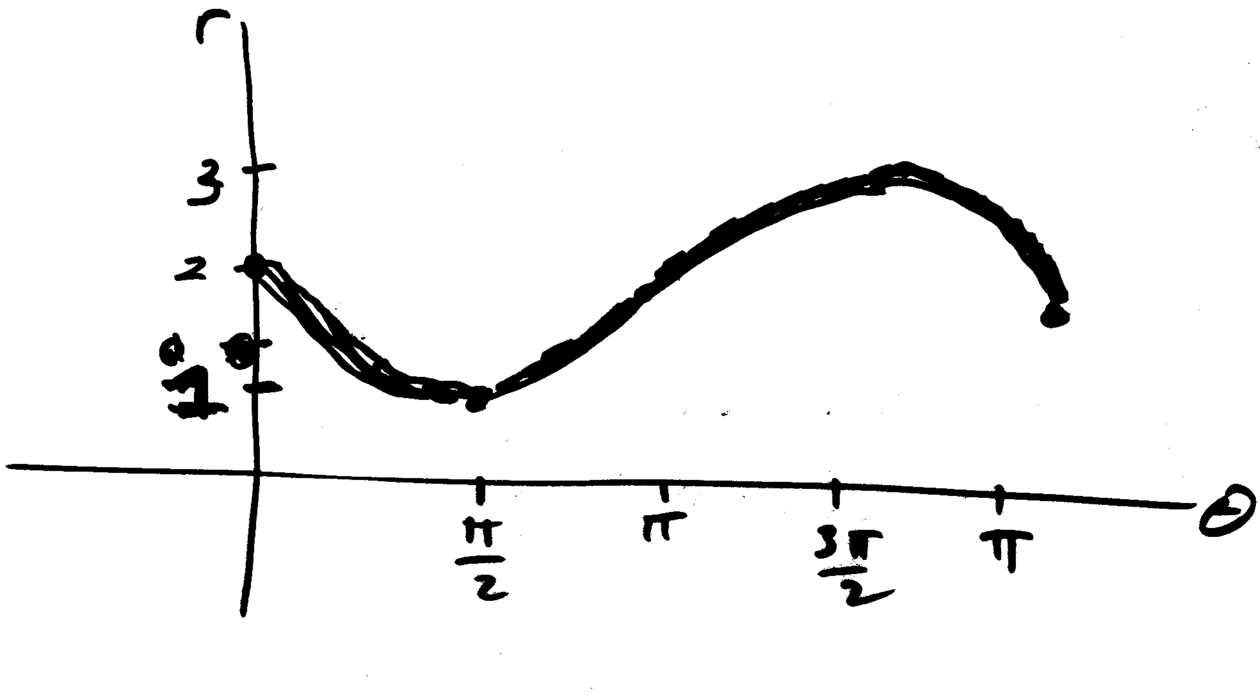
$r^2 = 4y$; But $r^2 = x^2 + y^2$;

So $\quad x^2 + y^2 = 4y \Longrightarrow$

$x^2 + y^2 - 4y = 0$
$x^2 + (y-2)^2 = 4$

Circle of radius 2, Center (0,2).

$r = 2 - \sin\theta$



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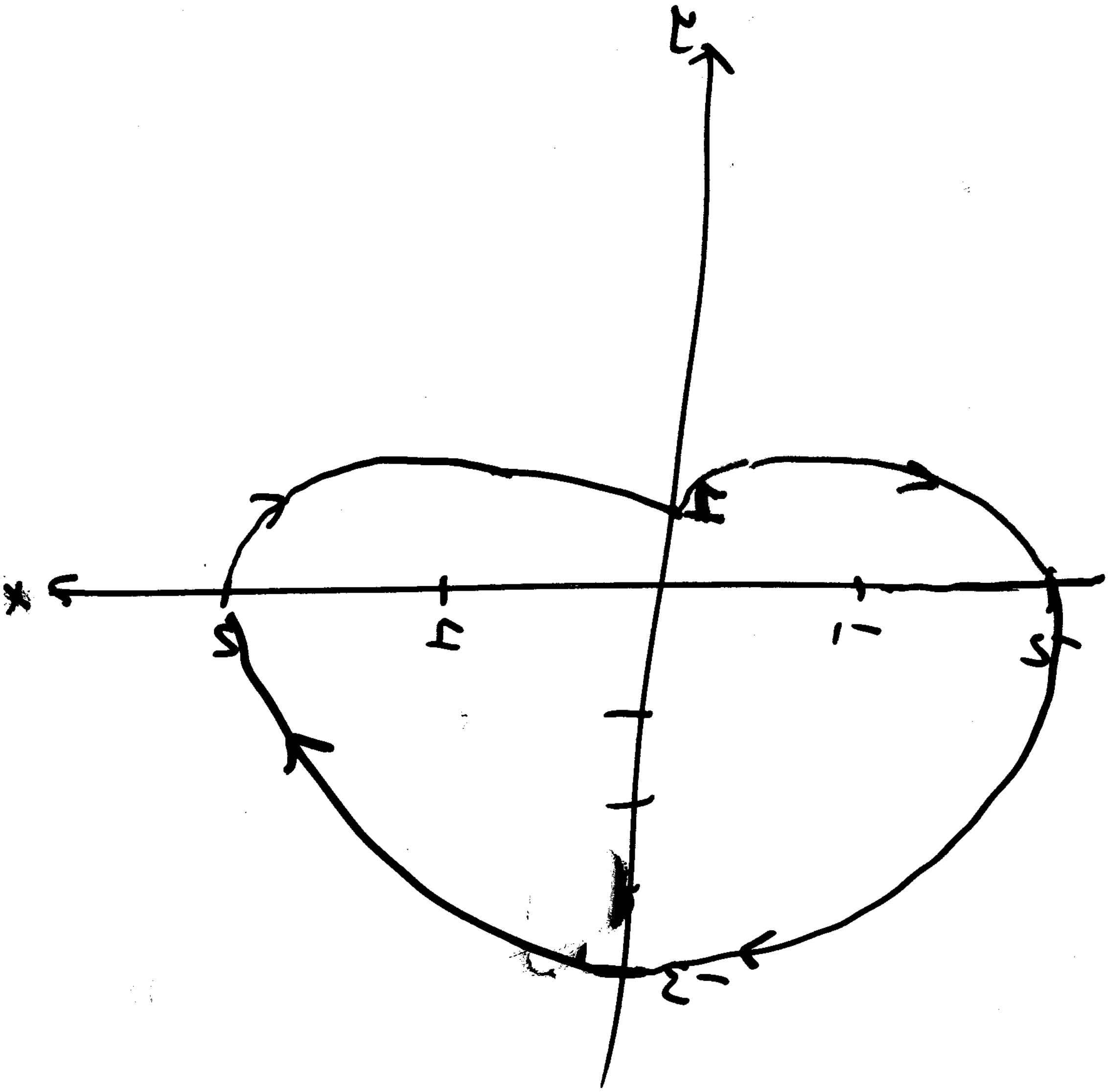

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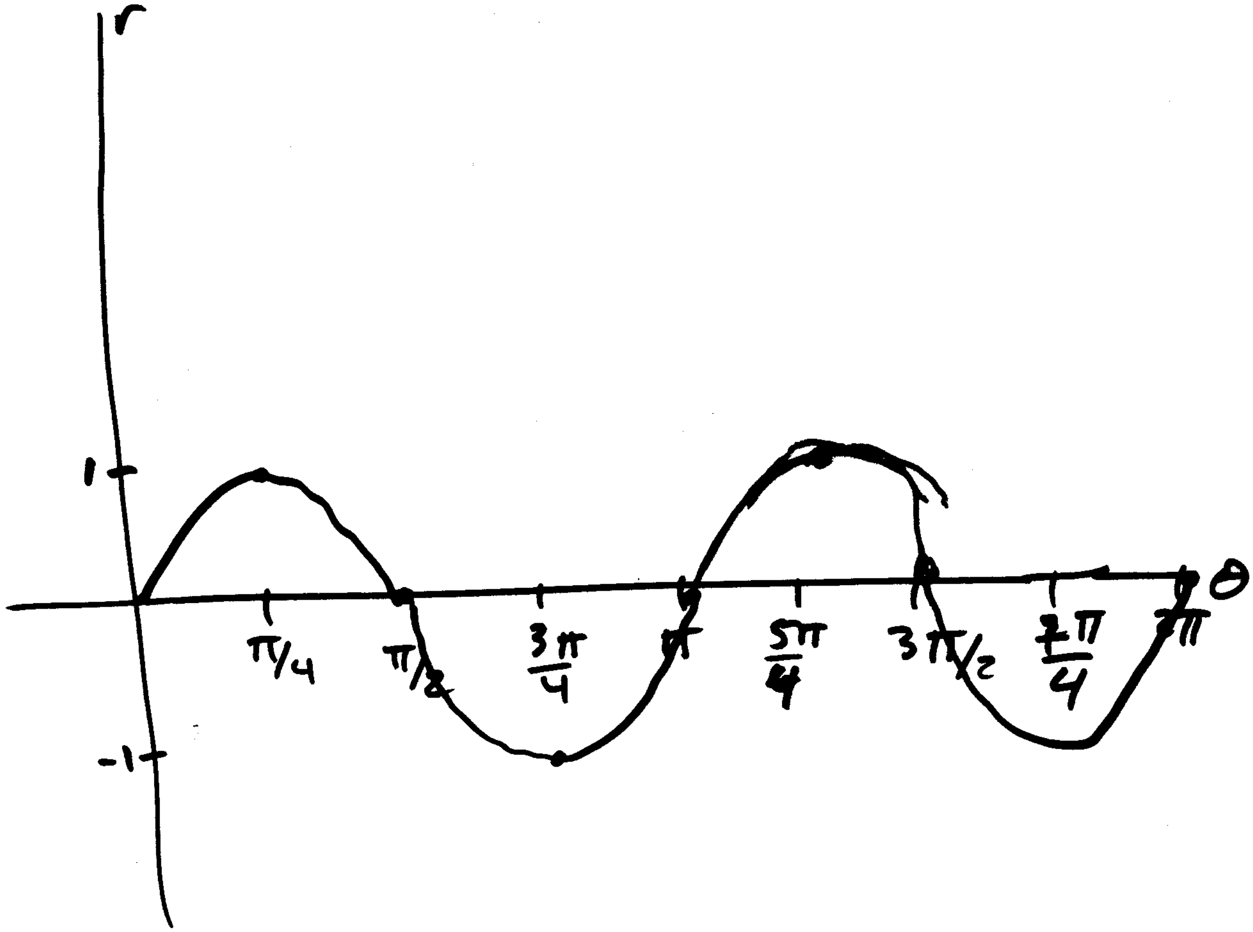
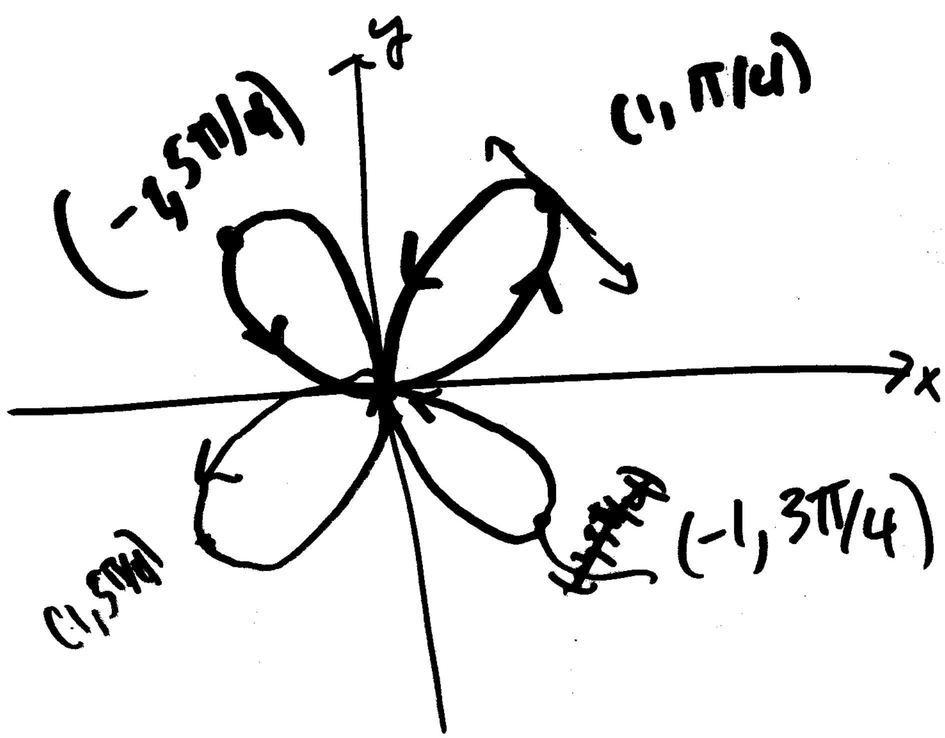
$r = \sin 2\theta$





$$= \frac{3\sqrt{3} - 4}{11}$$

Find the points where the Cardioid $r = 2 - \sin\theta$ has a horizontal tangent

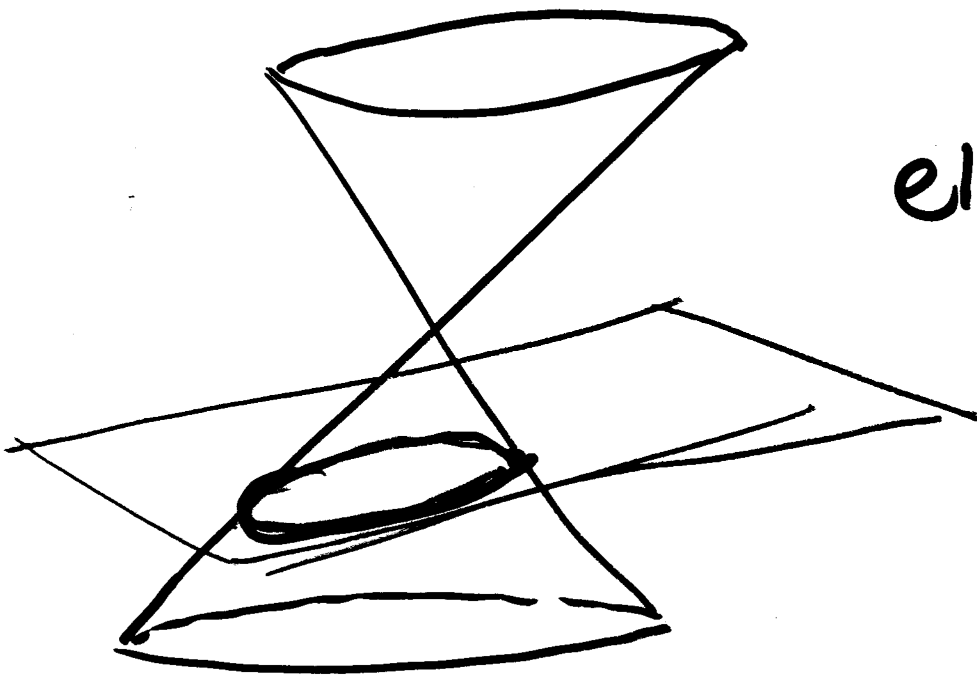
$$\frac{dy}{d\theta} = 2\cos\theta(1 - \sin\theta)$$

$\frac{dy}{d\theta} = 0$; $\cos\theta = 0$, or $\sin\theta = 1$

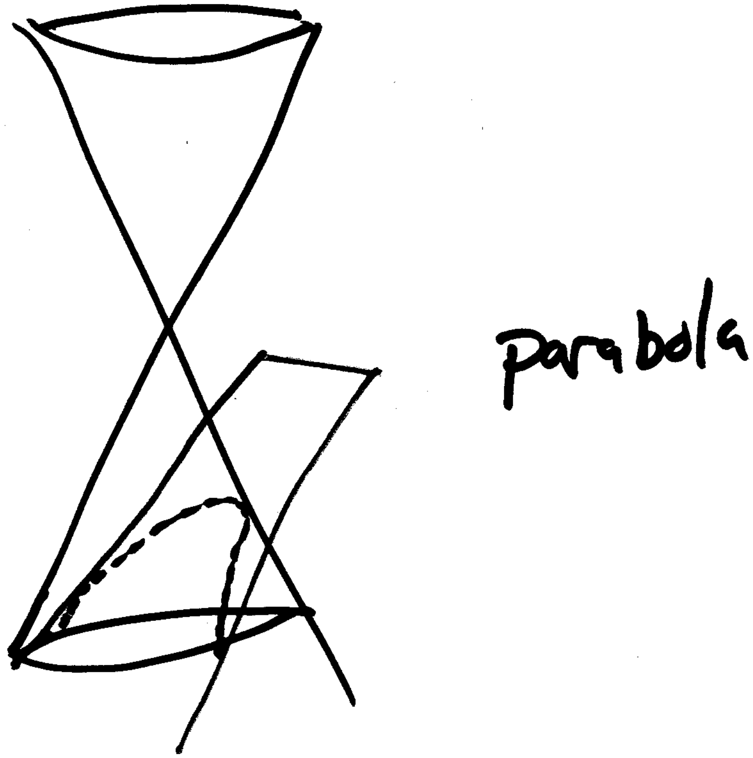
$\theta = \pi/2,\ 3\pi/2$

Conic Sections

A conic section is obtained by "Slicing" a cone by a plane



ellipse



Also hyperbolas (Monday)

Here we describe the conic sections by their properties and equations.

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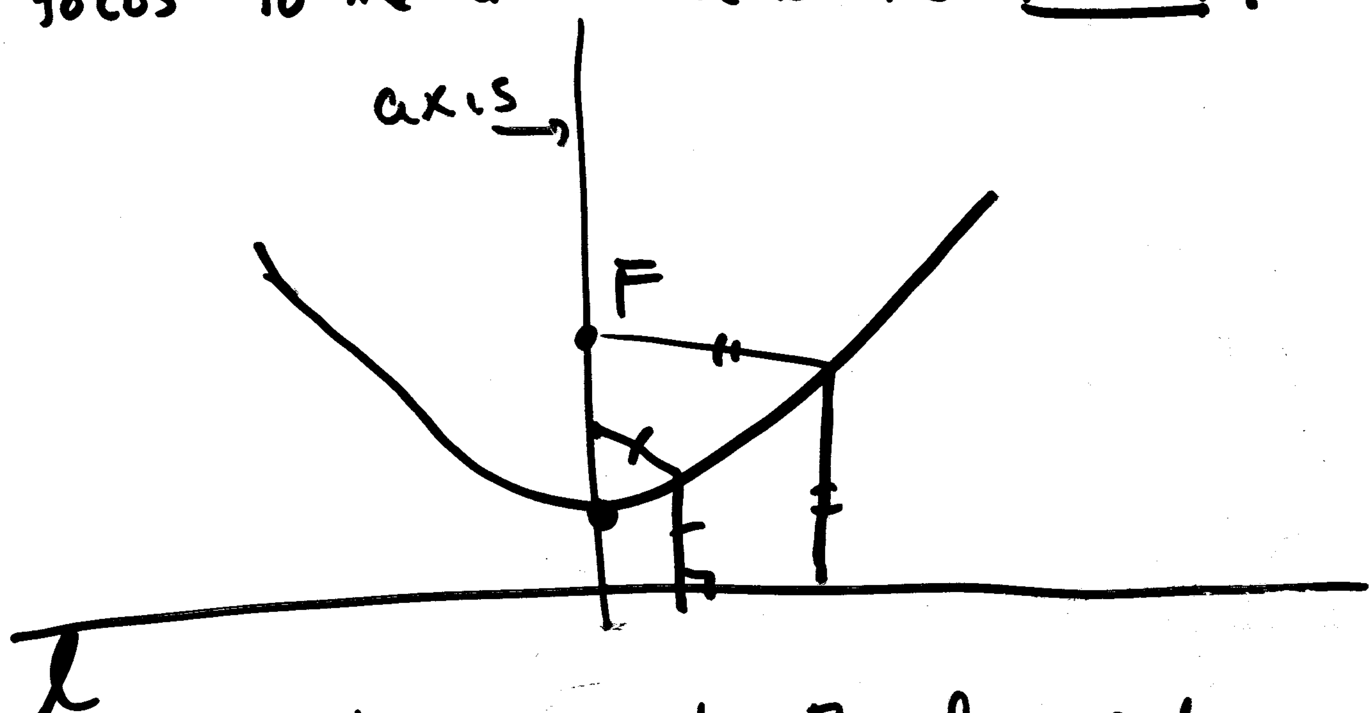
A parabola is the set of points in the plane which are <u>equidistant</u> from a fixed point F (focus) and a fixed line ℓ (directrix).

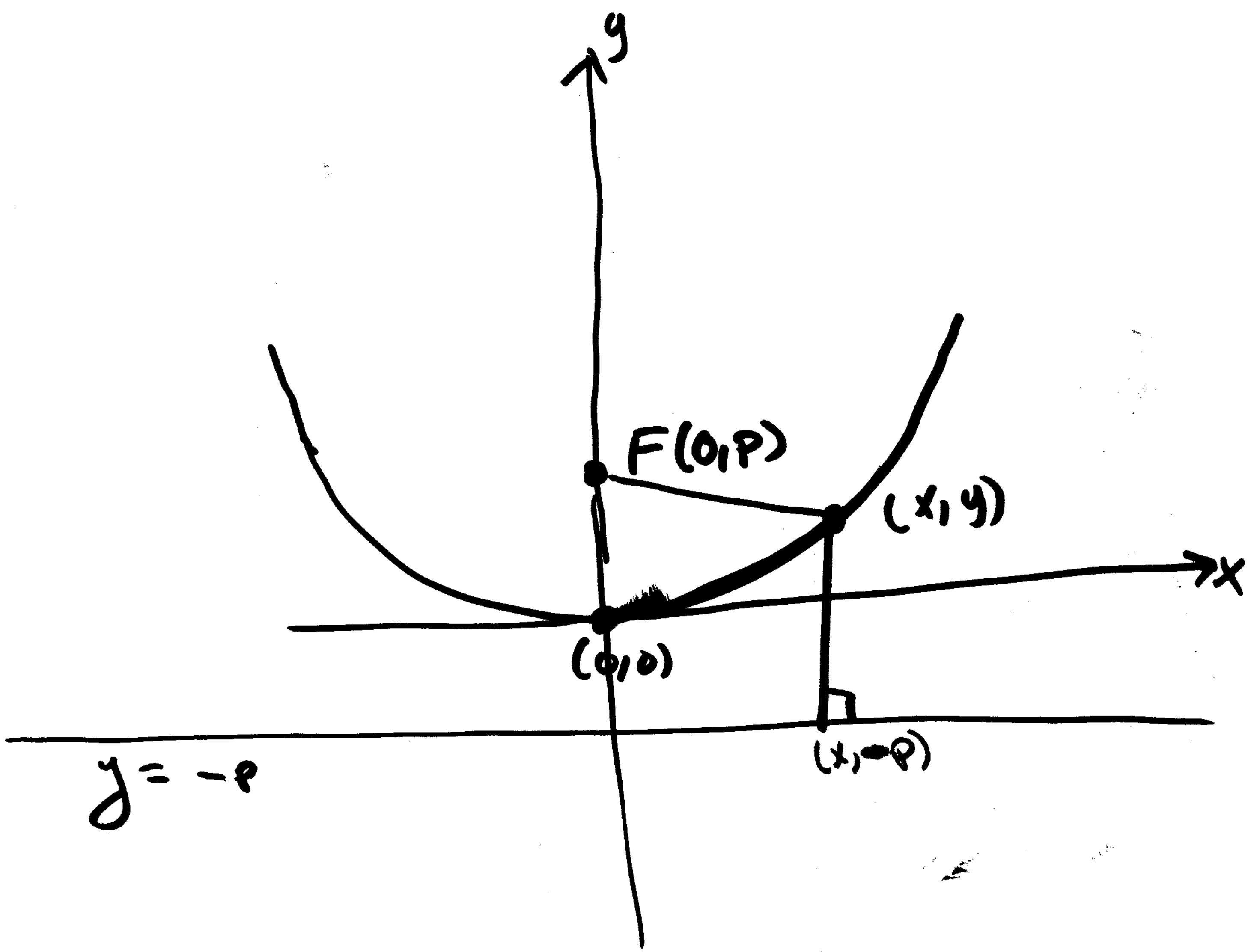
The midpoint of the segment from the focus to the directrx is the <u>vertex</u>.



The line connecting the focus and vertex is the <u>axis</u>

4 In order to write the equation of a parabola we first assume the vertex is $(0,0)$, and focus is $(0,p)$.

So the directrix is: ______________



5 The points P parabola satisfies

$$|FP| = |Pl|$$

If $P = (x, y)$, then $|FP| =$

$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

On the other hand $|Pl| = |y+p|$

So the equation of the parabola

is: $\sqrt{x^2 + (y-p)^2} = |y+p| \Rightarrow$

$$x^2 + (y-p)^2 = (y+p)^2$$

$$x^2 + \boxed{y^2} \boxed{-2py} + \boxed{p^2} = \boxed{y^2} \boxed{+2py} + \boxed{p^2}$$

$$\Rightarrow \quad x^2 = 4py$$

This is the equation of a parabola with focus $(0,p)$ and directrix $y = -p$.

We can rewrite this as:

$y = ax^2$, with $a = \frac{1}{4p}$

The parabola opens upward (concave up)

if $p > 0$ (i.e. if $a > 0$) and down

if $p < 0$. Note the graph must

be symmetric with respect to the y-axis.

If we interchange the roles of

y and x we get $\boxed{y^2 = 4px}$

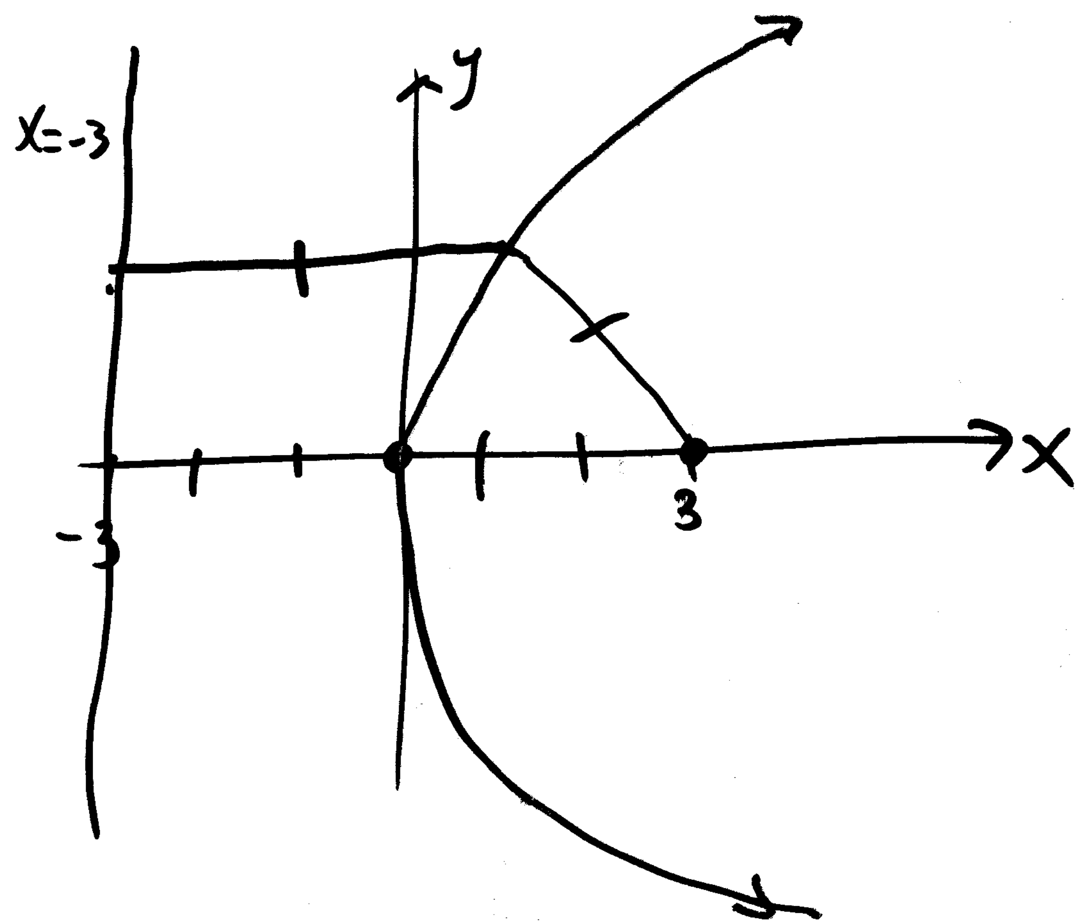
and this is the equation of a

parabola with focus $(p, 0)$ and

directrix $x = -p$.

<u>Example</u> Find the directrix, focus, and vertex of $y^2 = 12x$

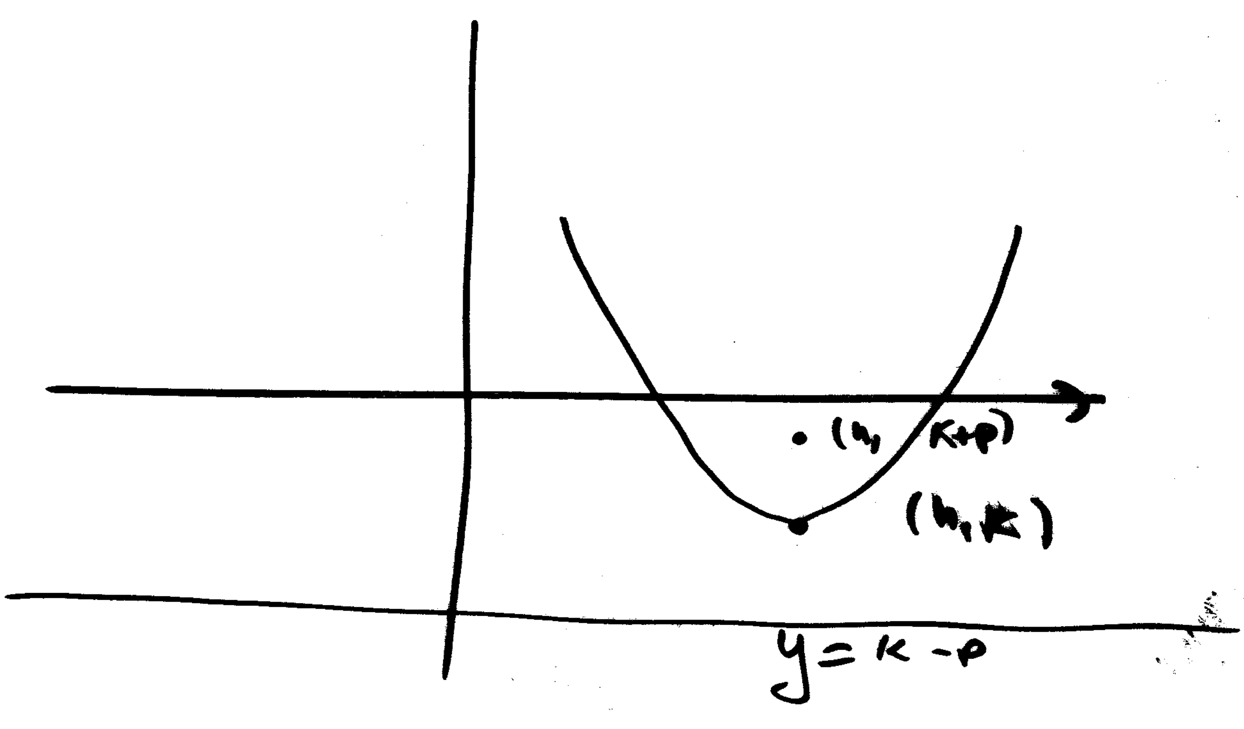
This is in the form $y^2 = 4px$, with $p = 3$. So the focus is $(3,0)$, directrix, $x = -3$, and vertex $(0,0)$



We can shift the focus and directrix in the "obvious" way.

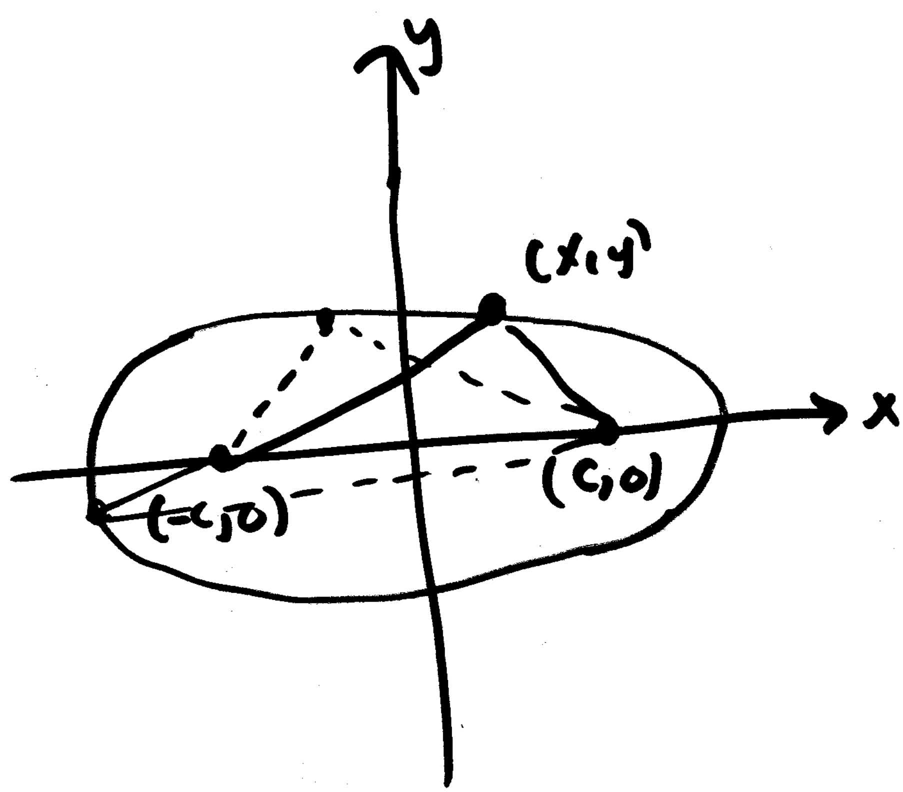
If the vertex is (h, k) and the directrix is $y = k-p$ then the focus is $(h, k+p)$ and the equation is:

$$(x-h)^2 = 4p(y-k)$$



An <u>ellipse</u> is the set of points in the plane THE SUM OF WHOSE DISTANCES FROM TWO FIXED POINTS F_1, F_2 (FOCI) IS A <u>CONSTANT</u>.

Let's again start with a simple case. Suppose the foci are at $(-c, 0)$ and $(c, 0)$.



10 Setting $F_1 = (-c, 0)$, $F_2 = (c, 0)$ and letting the constant distance which is the sum $|PF_1| + |PF_2|$ be $2a$ we get

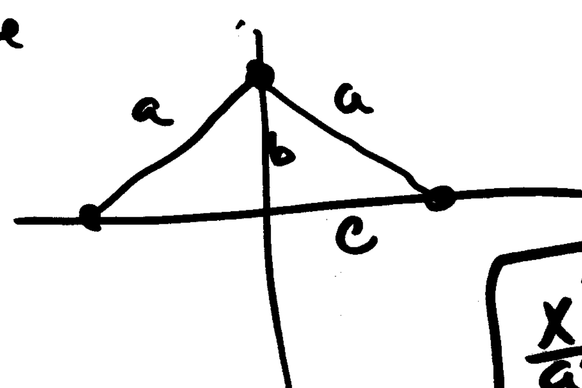
$$|PF_1| + |PF_2| = 2a \Rightarrow$$

$$\boxed{\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a}$$

This equation takes more work to simplify than that for a parabola (pg. 677).

Eventually we get: $(a^2 - c^2)x^2 + a^2 y^2 = a^2(a^2 - c^2)$

Note



$c < a$

Let $b^2 = a^2 - c^2$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

The equation

$$(2) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

with $a \geq b > 0$ is the equation of an ellipse with foci $(\pm c, 0)$, with $c^2 = a^2 - b^2$

Switching roles of (x,y) we see

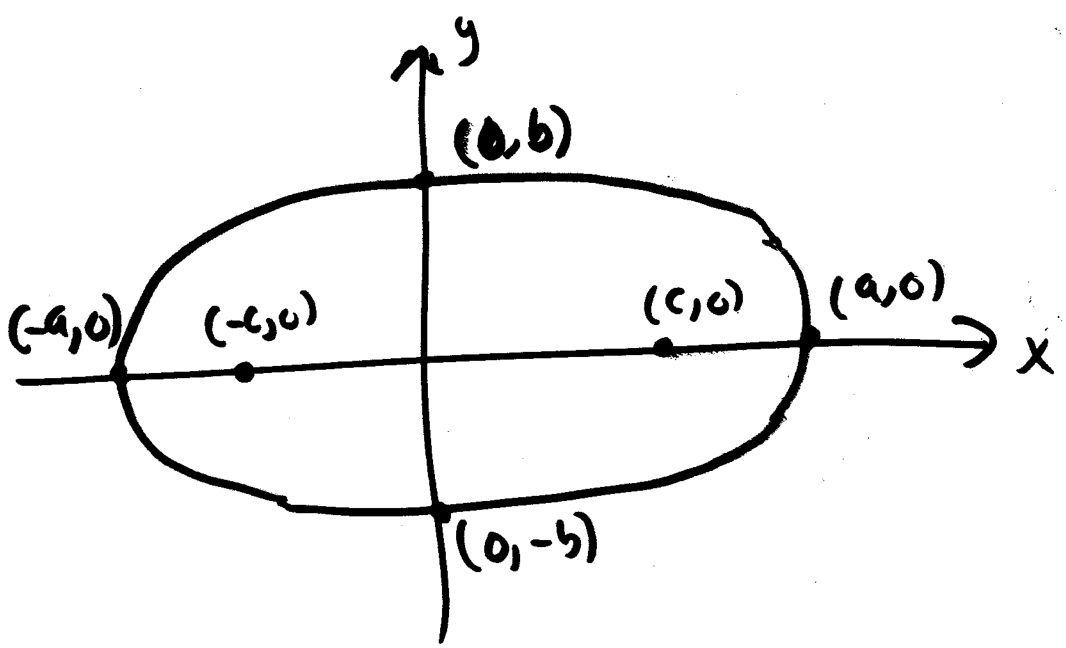
$$(3) \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a \geq b > 0$$

is that of an ellipse with foci $(0, \pm c)$, where $c^2 = a^2 - b^2$

For equation (2) the points $(-a,0)$ and $(a,0)$ are called the <u>vertices</u> of the ellipse

The line joining the vertices is the <u>major axis</u>.

(Similar for Eq. (3) with x, y roles switched)

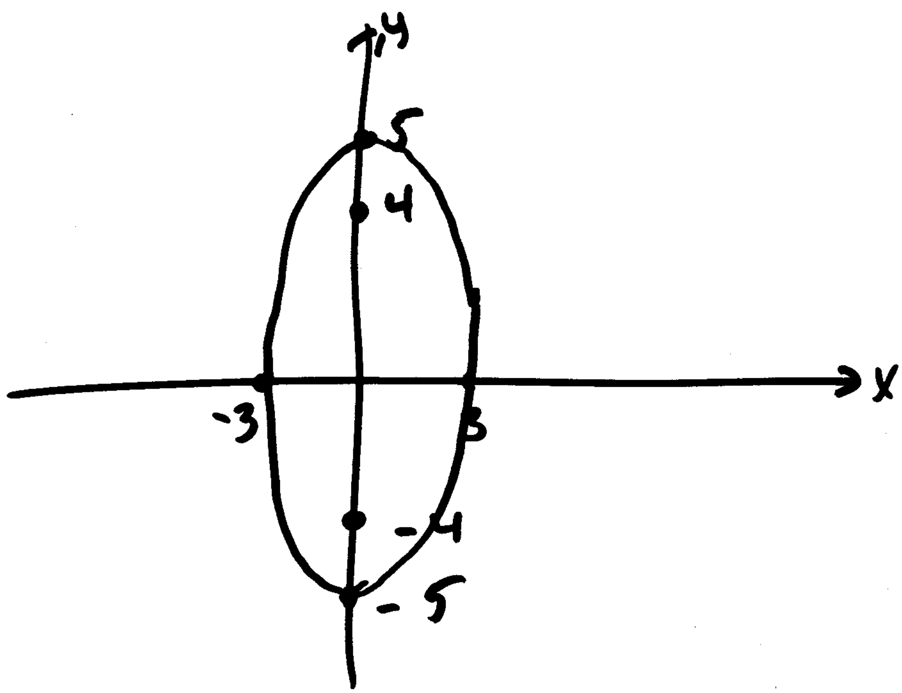


13 <u>Ex</u>: Find vertices, foci, and sketch: $25x^2+9y^2=225$

$$\frac{x^2}{9}+\frac{y^2}{25}=1;$$

$a=5$, $b=3$, $c^2=25-9=16$, $c=4$.

Foci : $(0,\pm 4)$; Vertices $(0,\pm 5)$



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Suppose we know the foci are $(\pm 5, 0)$, and vertices are $(\pm 13, 0)$.

What is the equation, and sketch?

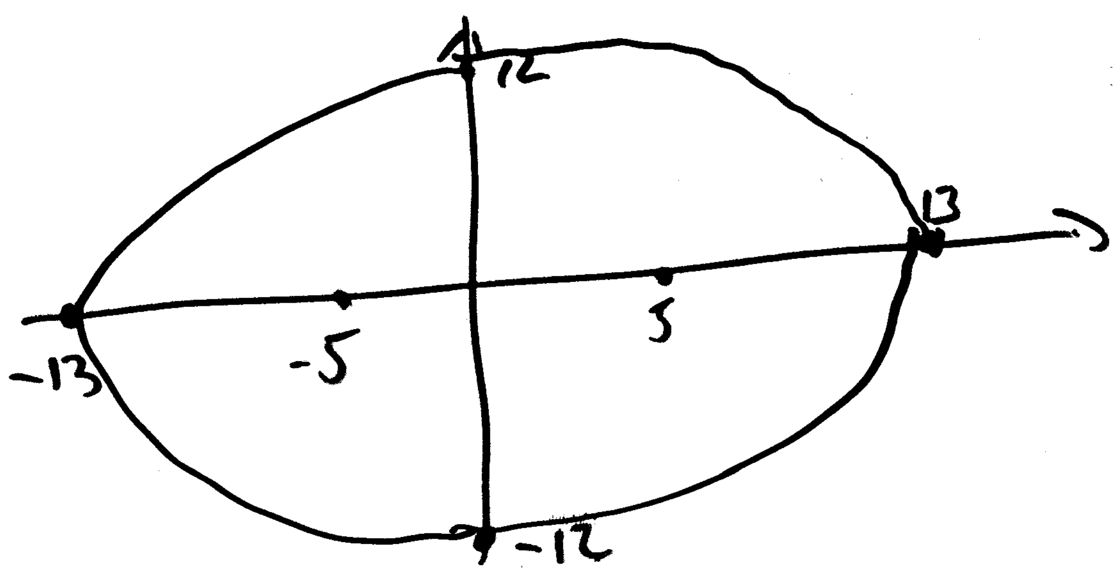
$c = 5$, $a = 13$, so

$b^2 = a^2 - c^2 = 13^2 - 5^2 = 12^2$

So,

$$\frac{x^2}{(13)^2} + \frac{y^2}{(12)^2} = 1$$

$$\frac{x^2}{169} + \frac{y^2}{144} = 1$$



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If the foci are:

$(h \pm c, k)$ and

vertices $(h \pm a, k)$, then

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$b^2 = c^2 - a^2.$