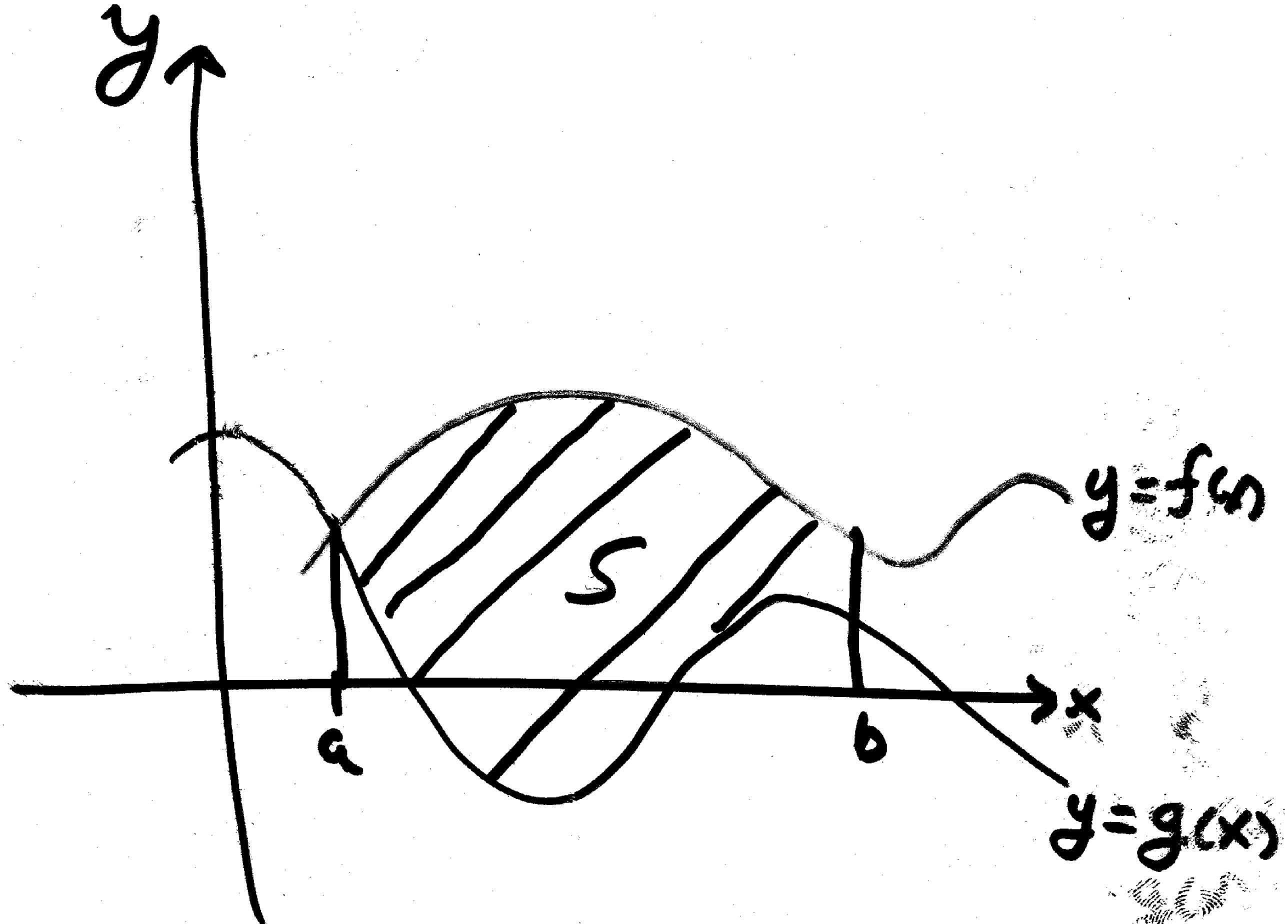
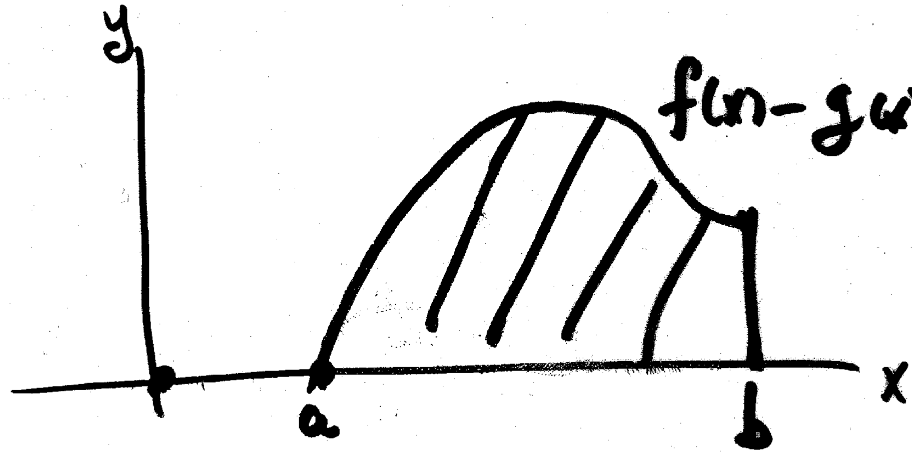
$\boxed{4.1}$ # Area between curves



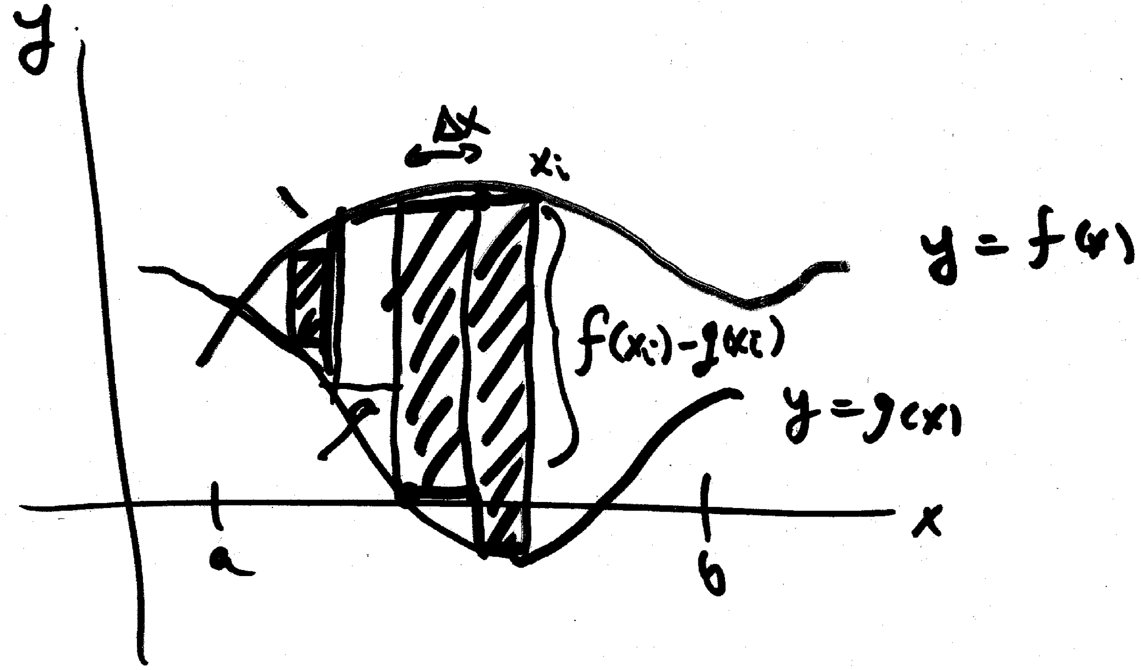
Suppose we want to find the area between the graphs of two functions



$\Rightarrow$ Assume $f(x) \geq g(x)$ for $a \leq x \leq b$

42 Simple analysis with intermediate

rectangles, as below



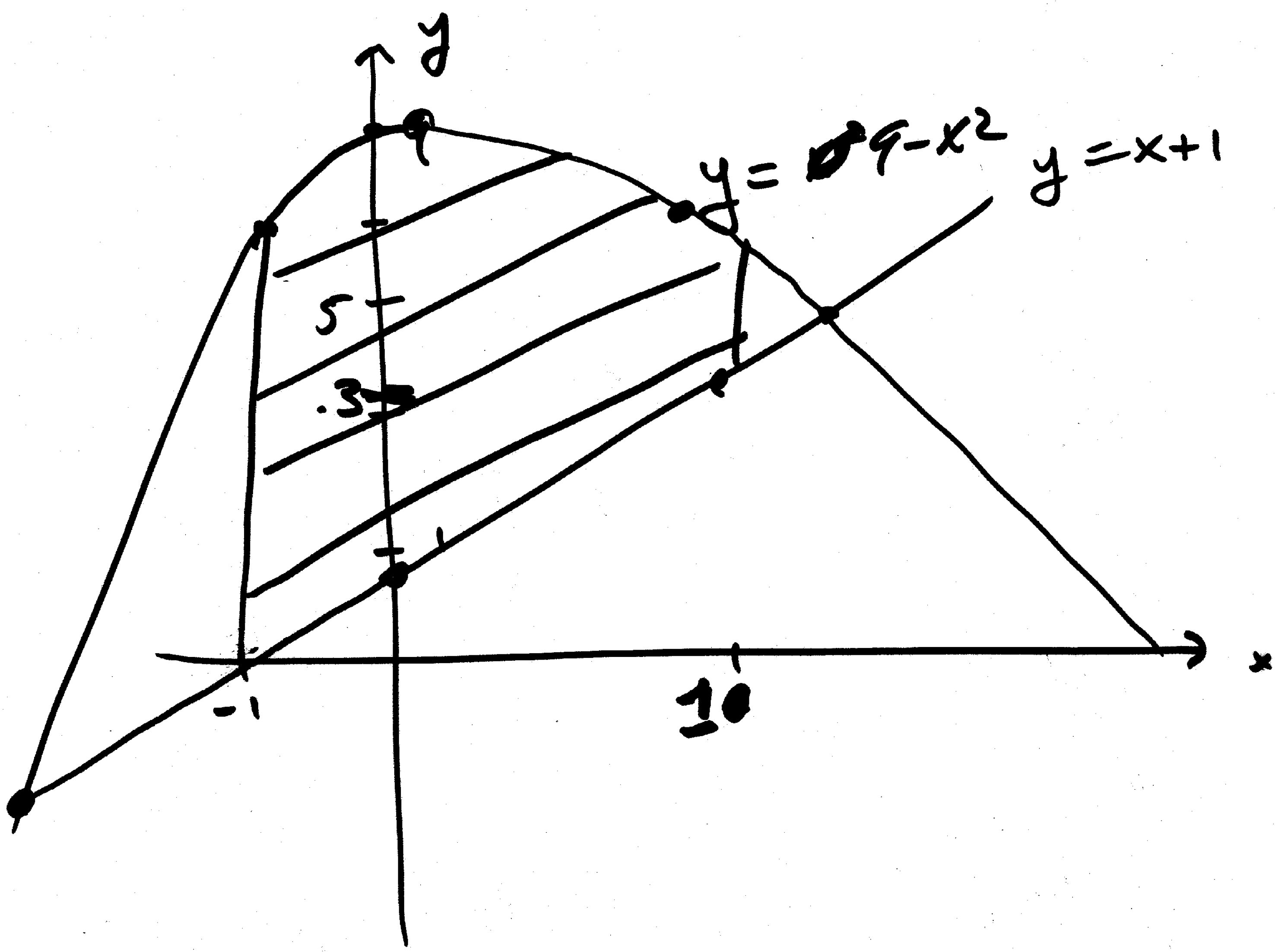
(letting rectangles have smaller and smaller to approximate area more and more closely)

$$A = \int_a^b (f(x) - g(x))\, dx$$

Ex: Find the area between

$y = x+1$, $y = 9-x^2$, between

$x = -1$, $x = 1$



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$$A = \int_{-1}^{2} \left((9-x^2) - (x+1)\right) dx$$

$$= \int_{-1}^{2} (8 - x - x^2)\, dx =$$

$$8x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-1}^{2} =$$

$$\left[16 - \frac{4}{2} - \frac{8}{3}\right] - \left[-8 - \frac{1}{2} - \frac{-1}{3}\right]$$

$$= \frac{127}{6}, \quad \text{sq. units.}$$

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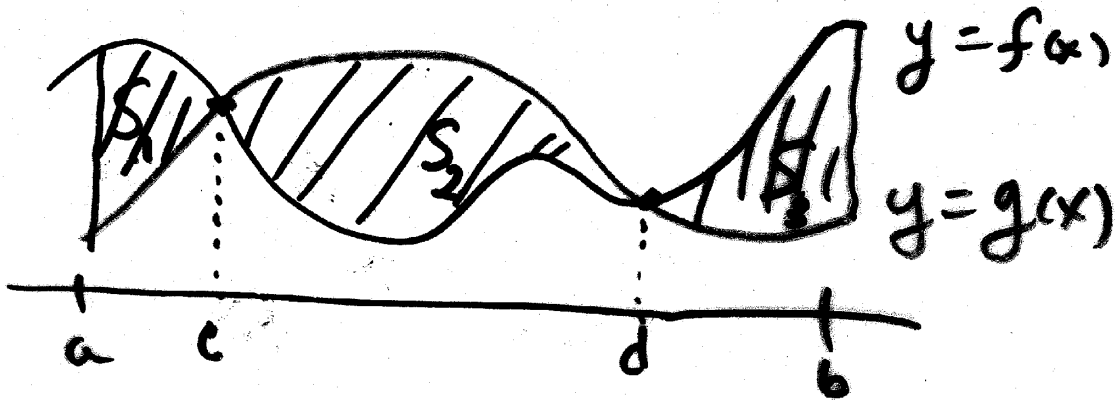
$$A = \int_{-1}^{2} \left((9 - x^2) - (x+1) \right) dx$$

$$= \int_{-1}^{2} (8 - x - x^2)\, dx = 8x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-1}^{2}$$

$$= \left[16 - 2 - 8/3 \right] - \left[-8 - \frac{1}{2} - \frac{-1}{3} \right]$$

$$= 37/2$$

Suppose the graphs of $y = f(x)$ and $y = g(x)$ intersect more than once in $[a, b]$. How do we find the area between the two curves?



Then we find the area between the curves, between each intersection.

In the figure this would be:

$$A = S_1 + S_2 + S_3 =$$

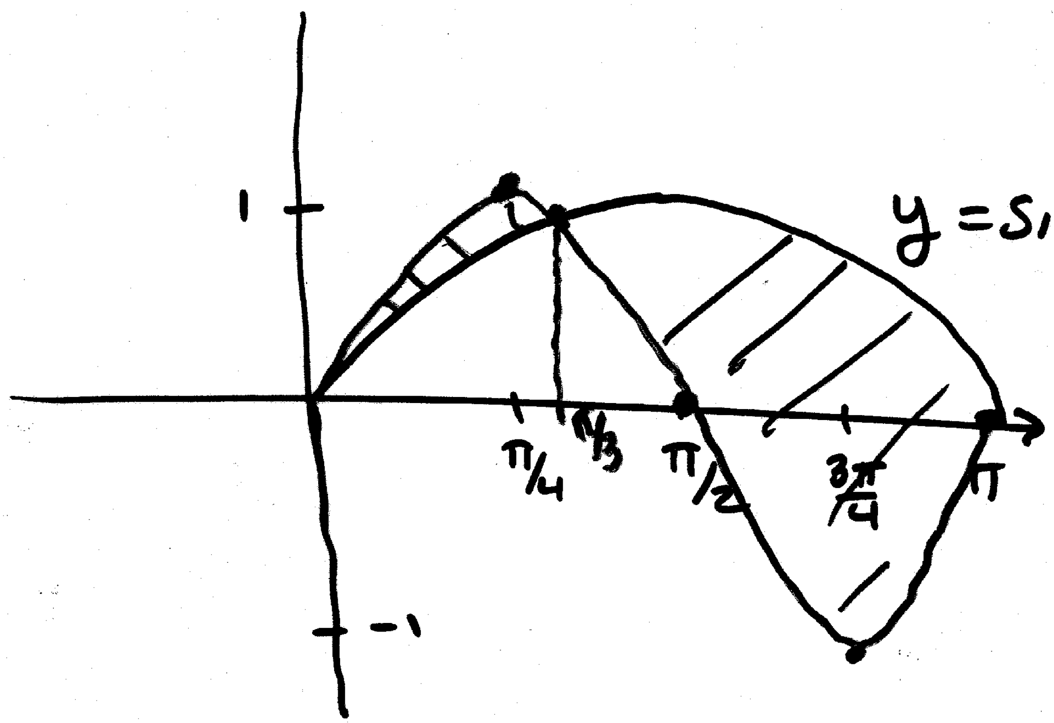
$$\int_a^c (f(x) - g(x))\,dx + \int_c^d (g(x) - f(x))\,dx +$$

$$\int_d^b (f(x) - g(x))\,dx.$$

In general we can write this as:

$$\boxed{A = \int_a^b |f(x) - g(x)|\,dx}$$

47 Find the area between $y = \sin x$ and $y = \sin 2x$ for $0 \le x \le \pi$.



<u>Soln:</u> Need to solve $\sin 2x = \sin x$.

Since $\sin 2x = 2\sin x \cos x$ this becomes

$$2\sin x \cos x = \sin x$$

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$\boxed{\text{Note this is } \underline{\underline{\text{not}}} \text{ just } \cos x = \frac{1}{2}}$

<u>EITHER</u> $\sin x = 0$ or

$\cos x = 1/2$.

For $0 \leq x \leq \pi$, this is

$x = 0, \frac{\pi}{3}, \pi$.

YOU ALSO NEED TO DETERMINE which of $\sin x$, $\sin 2x$ is greater on each interval. ONE WAY: PICK SOME POINT x_0 between 0 and $\pi/3$, and another x_1 between $\pi/3$ and π and see which is greater.

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Take $x_0 = \pi/4$. Then

$$\sin(x_0) = \sin \pi/4 = \frac{\sqrt{2}}{2}$$

$$\sin(2x_0) = \sin(\pi/2) = 1,$$

So for $0 \le x \le \pi/3$, $\sin 2x \geq \sin x$

Take $x_1 = \frac{\pi}{2}$,

$$\sin(x_1) = \sin(\pi/2) = 1$$

$$\sin 2x_1 = \sin(\pi) = 0$$

So for $\frac{\pi}{3} \le x \le \pi$, $\sin x \geq \sin 2x$.

Thus, $A = \displaystyle\int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\,dx$

$$= \left[-\frac{\cos 2x}{2} + \cos x\right]_0^{\pi/3} + \left[-\cos x + \frac{\cos 2x}{2}\right]_{\pi/3}^{\pi}$$

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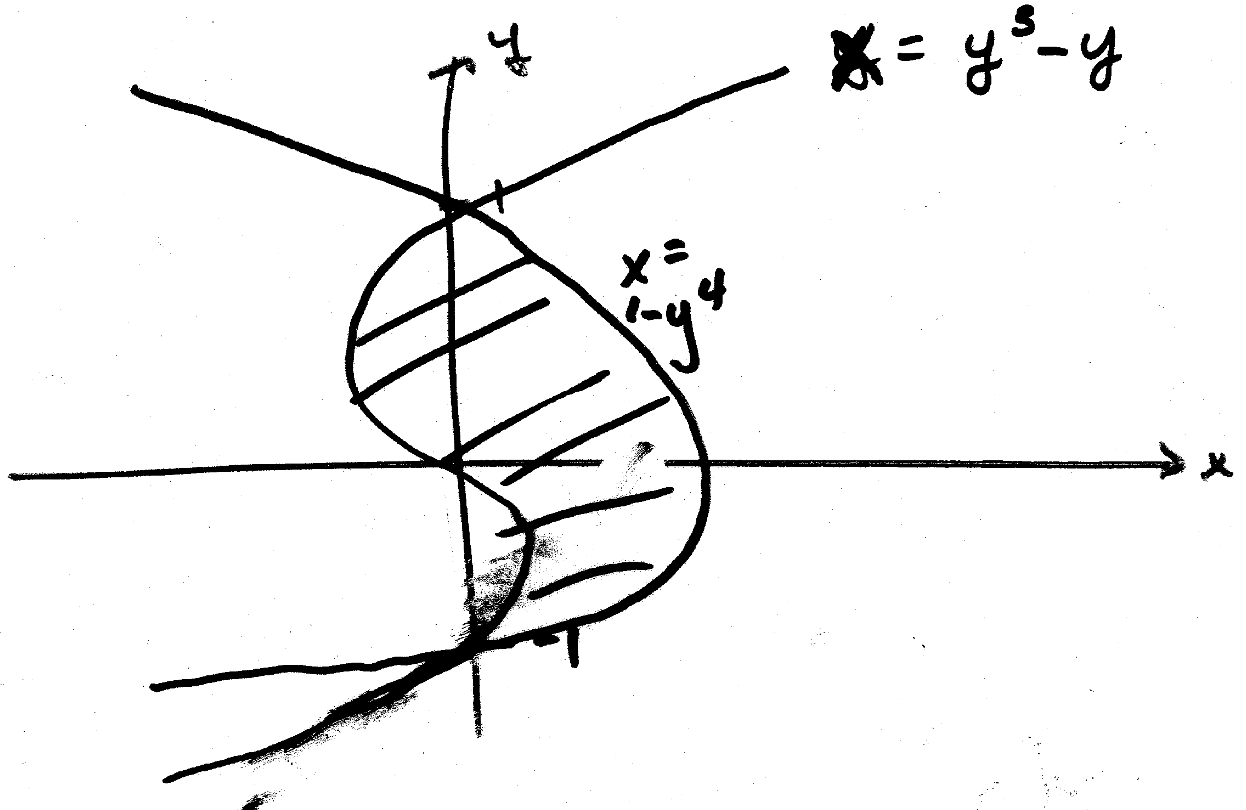
Sometimes easier to think of x as a function of y

Find the area between

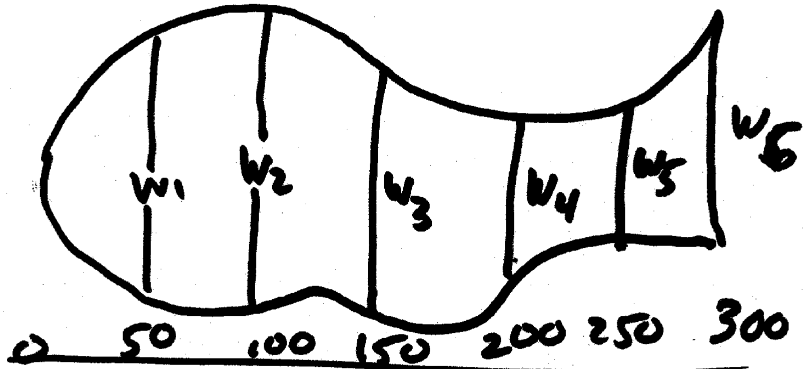
$x = y^3 - y$ and $x = 1 - y^4$



$$A = \int_{-1}^{1} \left((1-y^4) - (y^3-y) \right) dy \ \ldots$$

Can use midpoint approximations:

<u>EX</u>: Suppose we have an irregularly shaped tract of land as in the figure. Estimate the area.



Measurement in feet.

x	0	50	100	150	200	250	300
w	0	165	192	146	63	42	84

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Use the midpoint rule with mesh 100ft.

That is, replace actual area with rectangle with area $100 \cdot W_i$ for $i = 1, 3, 5$

$$A \approx 100 \cdot 165 + 100 \cdot 146 + 42 \cdot 100$$
$$= 100(165 + 146 + 42) = 100(353) = 35{,}300 \text{ sq. ft.}$$

So Area is approximately 35,300 sq. ft.