Integration by parts

$$\int x \sin x \, dx = ?$$

Find a function $F(x)$ with

$F'(x) = x \sin x$?

Might guess that a choice of $F(x)$ would be:

$F(x) = -x\cos x$. But then

$$\frac{d}{dx}(-x\cos x) = -\underline{\cos x} + \underline{x \sin x}$$

What can we do about this term?

Instead choose $F(x) = -x\cos x + \underline{\sin x}$

Then $F'(x) = (-\cos x + x\sin x) + \cos x$

$$= x\,\overset{\sin x}{\cancel{\cos x}}$$

Here we used the product rule:

$$\circledast \quad (fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x).$$

We'd like to be more systematic:

Rewrite $\circledast$:

$$(fg)'(x) - f'(x)\,g(x) = f(x)\,g'(x).$$

Now take integral of each side and use the fundamental Thm of Calc.

$$f(x)\,g(x) - \int f'(x)\,g(x)\,dx = \int f(x)\,g'(x)\,dx$$

So if we see

$$\int f(x)\,g'(x)\,dx$$ we

can write:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

This is known as Integration by Parts

OFTEN WRITTEN:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

For $\int x \sin x \, dx$, let $u = x$ $\quad du = dx$

$dv = \sin x \, dx$ $\quad v = -\cos x$

$$\int x \sin x \, dx = -x\cos x - \int -\cos x \, dx$$

$$= -x\cos x + \sin x + C$$

4/ Choosing which **part** is "u" and which is "dv" is important. Choose so the new integral is simpler.

Ex: $\int \ln x \, dx$; $u = \ln x \quad du = \frac{1}{x} dx$

$dv = dx \quad v = x$

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$$
$$= x \ln x - x + C.$$

What about $\int \underline{x}^2 \underline{\sin x} \, dx$?

Take $u = x^2 \quad du = 2x\,dx$

$dv = \sin x \, dx \quad v = -\cos x$

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So

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int -2x \cos x \, dx$$
$$= -x^2 \cos x + 2\int x \cos x \, dx$$

Need to use parts again:

$z = x \qquad dz = dx$

$dw = \cos x \, dx \quad w = \sin x$

$$\int x^2 \sin x \, dx =$$

$$-x^2 \cos x + 2x \sin x - 2\int \sin x \, dx$$

$$= 2x \sin x - x^2 \cos x + 2\cos x + C$$

What about:

$$\int \sin^{-1}x \, dx \ ?$$

$u = \sin^{-1}x \qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \qquad v = x$

$$\int \sin^{-1}x \, dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Now use: $\underline{w = 1-x^2}$ $\quad dw = -2x\,dx$

So $\int \sin^{-1}x \, dx = x\sin^{-1}x + \frac{1}{2}\int \frac{dw}{\sqrt{w}}$

$=$

$= x\sin^{-1}x + \frac{1}{2}\left(2 \cdot w^{1/2}\right) + C$

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Find $\int e^{-\theta} \cos\theta \, d\theta$

$u = e^{-\theta}$ $\qquad$ $du = -e^{-\theta} d\theta$

$dv = \cos\theta \, d\theta$ $\qquad$ $v = \sin\theta$

$$\int e^{-\theta} \cos\theta \, d\theta = e^{-\theta} \sin\theta - \int -e^{-\theta} \sin\theta \, d\theta$$

$$= e^{-\theta} \sin\theta + \int e^{-\theta} \sin\theta \, d\theta$$

$z = e^{-\theta}$ $\qquad$ $dz = -e^{-\theta} d\theta$

$dw = \sin\theta \, d\theta$ $\qquad$ $w = -\cos\theta$.

(i) $\int e^{-\theta}\cos\theta \, d\theta =$ (Use $u = e^{-\theta}$, $dv = \cos\theta \, d\theta$

$e^{-\theta}\sin\theta + \int e^{-\theta}\sin\theta \, d\theta =$

$e^{-\theta}\sin\theta + -e^{-\theta}\cos\theta - \int(-e^{-\theta})(-\cos\theta) \, d\theta$

$= e^{-\theta}\sin\theta - e^{-\theta}\cos\theta - \int e^{-\theta}\cos\theta \, d\theta$

$2\int e^{-\theta}\cos\theta = e^{-\theta}\sin\theta - e^{-\theta}\cos\theta + C'$

$\int e^{-\theta}\cos\theta \, d\theta = \frac{e^{-\theta}}{2}(\sin\theta - \cos\theta) + C'$

use $\underline{u = \cos\theta}$ $dv = e^{-\theta} \, d\theta$

$\int e^{-\theta}\cos\theta \, d\theta = -e^{-\theta}\cos\theta - \int -e^{-\theta}(-\sin\theta) \, d\theta$

$= -e^{-\theta}\cos\theta - \int e^{-\theta}\sin\theta \, d\theta =$

$-e^{-\theta}\cos\theta - \left[-e^{-\theta}\sin\theta - \int -e^{-\theta}\cos\theta \, d\theta\right]$

$= e^{-\theta}(\sin\theta - \cos\theta) - \int e^{-\theta}\cos\theta \, d\theta$

so again:

$\int e^{-\theta}\cos\theta \, d\theta = \frac{e^{-\theta}}{2}(\sin\theta - \cos\theta) + C$

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How about:

$$\int x\, 5^x \, dx$$

$u = x$ $\quad du = dx$

$dv = 5^x dx$ $\quad V =$

$$5^x = e^{\ln 5^x} = e^{x \cdot \ln 5}$$

$$\int e^{8x} dx = \frac{1}{8} e^{8x} + C$$

$$\int 5^x dx = \frac{1}{\ln 5} \cdot e^{\ln 5 \cdot x} + C = \frac{5^x}{\ln 5} + C$$

$$\int e^{\ln 5 \cdot x} = \frac{1}{\ln 5} \cdot e^{\ln 5 \cdot x} + C$$

$$= 5^x / \ln 5 + C$$