Trigonometric Integrals

We now consider how to integrate trig functions, such as

$$\int \sin^5 x \, dx$$

Here we can do the following:

Write

$$\int \sin^5 x \, dx = \int \sin^4 x \cdot \underline{\sin x \, dx}$$

Now use $\sin^4 x =$

$$(\sin^2 x)^2 = (1 - \cos^2 x)^2$$

2)

So

$$\int \sin^5 x\, dx = \int (1-\cos^2 x)^2 \sin x\, dx$$

Now substitute

$u = \cos x$, $du = -\sin x\, dx$

$$\int \sin^5 x\, dx = \int (1-u^2)^2 (-du)$$

$$= -\int (1-2u^2+u^4)\, du =$$

$$-\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C =$$

$$-\left(\cos x - \frac{2}{3}\cos^3 x + \frac{\cos^5 x}{5}\right) + C$$

What about

$$\int \cos^3 x \sin^2 x \, dx.$$

If we again write

$\sin^2 x = 1 - \cos^2 x$, we will have

no "$\sin x \, dx$" term left for

the substitution "$u = \cos x$."

Instead ~~take~~ write:

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \, \underline{\cos^2 x} \, \underline{\cos x \, dx}$$

$$= \int \sin^2 x \, (1 - \sin^2 x) \cos x \, dx$$

Now substitute $u = \sin x$

$du = \cos x \, dx$

4 so

$$\int \sin^2 x \cos^3 x \, dx =$$

$$\int u^2(1-u^2)\, du = \int (u^2 - u^4)\, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

STRATEGY FOR $\int \sin^m x \cos^n x \, dx$

(i) If $m = 2k+1$ is odd:
write $\sin^m x = \sin^{2k} x \sin x = (1-\cos^2 x)^k \sin x$ [redacted] and substitute
$u = \cos x$, $du = -\sin x \, dx$

(ii) If $n = 2k+1$ is odd, write
$\cos^n x = (1-\sin^2 x)^k \cos x$ [redacted] and
substitute $u = \sin x$, $du = \cos x \, dx$

(iii) If n, m are both even: use

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

and $\sin x \cos x = \frac{1}{2} \sin 2x$.

6

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1+\cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2} \right) dx$$

$$= \frac{1}{4} \left[\frac{3}{2}x + \sin 2x + \frac{\sin 4x}{8} \right] + C'$$

7]

$$\int \tan^4 x \sec^4 x \quad dx \ ?$$

Recall:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

and:

$$\tan^2 x + 1 = \sec^2 x$$

So write

$$\int \tan^4 x \sec^4 x \, dx =$$

$$\int \tan^4 x \sec^2 x \cdot \sec^2 x \, dx =$$

$$\int \tan^4 x \,(1+\tan^2 x)\cdot \sec^2 x \, dx$$

Let $u = \tan x$, $\quad du = \sec^2 x \, dx$

8

$$\int \tan^4 x \sec^4 x \, dx =$$

$$\int u^4 (1+u^2)\, du = \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

How about $\int \tan^3 x \sec^3 x \, dx$

Now write

$$\int \tan^3 x \sec^3 x = \int \tan^2 x \cdot \sec^2 x \, (\sec x \tan x)\, dx$$

$$= \int (\sec^2 x - 1) \sec^2 x \cdot \sec x \tan x \, dx,$$

Let $u = \sec x$ $\quad du = \sec x \tan x \, dx$

9)

$$\int \tan^3 x \sec^3 x \, dx =$$

$$\int (u^2-1)u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

STRATEGY FOR

$$\int \sec^n x \tan^m x \, dx$$

(i) IF $n \overset{=2K}{}$ is even, write

$\sec^n x = (\sec^2 x)^{K-1} \sec^2 x =$

$(\tan^2 x + 1)^{K-1} \sec^2 x$, and substitute

$u = \tan x$, $\quad du = \sec^2 x \, dx$

(ii) If $m = 2K+1$ is odd : (and $n \geq 1$)

Write $\tan^m x = (\tan^2 x)^K \tan x$

$= (\sec^2 x - 1)^K \tan x$ and let $u = \sec x$,

$du = \sec x \tan x \, dx$

OTHER CASES ??

Need to use:

$$\int \tan x \, dx = \ln|\sec x| + C$$

and

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

(See derivation on pg. 480)

Note on definite integrals:
Remember to substitute for limits.

Ex: $\displaystyle\int_0^{\pi/3} \tan^5 x \sec x \, dx$

12

$$\int_0^{\pi/3} \tan^5 x \sec x \, dx =$$

$$\int_0^{\pi/3} \tan^4 x \tan x \sec x \, dx =$$

$$\int_0^{\pi/3} (\sec^2 x - 1)^2 \sec x \tan x \, dx$$

Let $u = \sec x$ $\quad du = \sec x \tan x \, dx$

$x=0, \; u=1 \; ; \quad x = \pi/3, \; u = 2$

Then,

$$\int_0^{\pi/3} \tan^5 x \sec x \, dx = \int_1^2 (u^2-1)^2 \, du$$

$$= \int_1^2 (u^4 - 2u^2 + 1) \, du$$

$$= \frac{u^5}{5} - \frac{2}{3}u^3 + u \Big|_1^2 = \ldots$$

13 (i) $\int \sin(mx)\cos(nx)\,dx$ or

(ii) $\int \sin(mx)\sin(nx)\,dx$ or

(iii) $\int \cos(mx)\cos(nx)\,dx$:

USE:

(i) $\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$

(ii) $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

(iii) $\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$

Ex: $\int \sin 5x \cos 3x\,dx =$ (use (i))

$\frac{1}{2}\int (\sin 2x + \sin 8x)\,dx = -\frac{1}{2}\left(\frac{\cos 2x}{2} + \frac{\cos 8x}{8}\right) + C$

TRIGONOMETRIC SUBSTITUTION

Evaluate

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx$$

POSSIBLE SUBSTITUTION?

$u = x^2+9$ $\quad du = 2x$?

or

$u =$ $\quad du =$?

NONE OF THESE WORK WELL

<u>INSTEAD:</u> TRY TO THINK OF A DIFFERENT SUBSTITUTION

$x = \_\_\_\_\_\_\_\_$??

2| Integrals involving: $\left(\begin{array}{l} a \text{ a constant} \\ x \text{ the variable} \end{array} \right)$

$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or

$\sqrt{x^2-a^2}$ can be dealt

with by <u>trigonometric</u> <u>substitution</u>

which is an example of

<u>inverse</u> <u>substitution</u>.

IF you SEE	SUBSTITUTE	Why?
$\sqrt{a^2-x^2}$	$x = a\sin\theta$, $\underline{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$, $-\pi/2 < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $0 \le \theta < \pi/2$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE:

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx \quad , \quad a=4,$$

Let $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$

$-\pi/2 \le \theta \le \pi/2$

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = \int \frac{4\cos\theta\, d\theta}{(4\sin\theta)^2\sqrt{16-(4\sin\theta)^2}}$$

$$= \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\ 4\sqrt{1-\sin^2\theta}}\ d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\ \sqrt{\cos^2\theta}}\ d\theta$$

NOTE: IN GENERAL $\sqrt{a^2} \neq a$

but $\sqrt{a^2} = |a|$

BUT OUR CHOICE $-\pi/2 \le \theta \le \pi/2$,

gives $\cos\theta \geq 0$, so $\sqrt{\cos^2\theta} = \cos\theta$

4

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx =$$

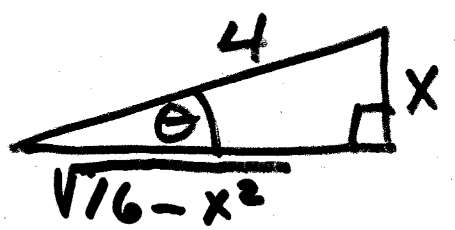
$$\frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{16}\int \csc^2\theta\,d\theta$$

$$= \frac{1}{16} -\cot\theta + C.$$

OUR ANSWER SHOULD BE IN TERMS OF $\underline{\underline{\underline{x}}}$

$x = 4\sin\theta$

$\sin\theta = x/4$



$$\cot\theta = \frac{-\sqrt{16-x^2}}{x}$$

SO $$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = \frac{-1}{16}\,\frac{\sqrt{16-x^2}}{x} + C$$

5

OUR EXAMPLE

$$\int \frac{x^3}{\sqrt{9+x^2}}\, dx \qquad (1)$$

$x = 3\tan\theta \qquad dx = 3\sec^2\theta\, d\theta$

θ RANGE: $-\pi/2 < \theta < \pi/2$

$$(1) = \int \frac{(3\tan\theta)^3\, 3\sec^2\theta\, d\theta}{\sqrt{9+(3\tan\theta)^2}}$$

$$= 27\int \frac{\tan^3\theta \cdot \sec^2\theta}{\sqrt{\tan^2\theta+1}}\, d\theta = \int \tan^3\theta\ \mathrm{Se}$$

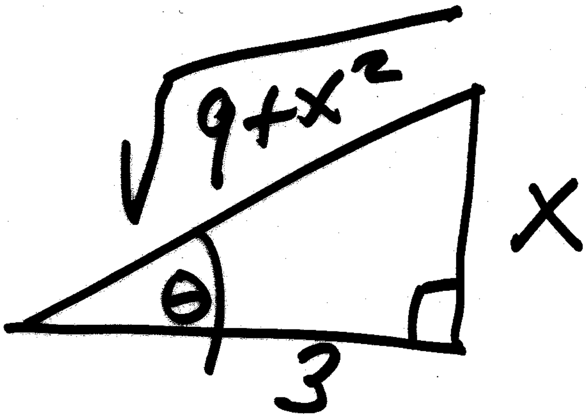
NEED TO USE TRIG. INTEGRAL METHOD

$u = \sec\theta \quad du = \sec\theta\tan\theta\, d\theta$

$$(1) = 27\int (u^2-1)\, du = \frac{u^3}{3} - u + C$$

$$= 27\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$

6|



$$\sec\theta = \frac{\sqrt{9+x^2}}{3}$$

So $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx =$

$$27\left(\frac{\sqrt{9+x^2}}{3}\right)^3 - 27\,\frac{\sqrt{9+x^2}}{3}$$

HOW ABOUT:

$$\int_5^{10} \frac{1}{x\sqrt{\cancel{25}^2 - \cancel{x}\,25}}\,dx \qquad \int_5^{10} \frac{1}{x\sqrt{x^2-25}}\,dx$$

SET $x = 5\sec\theta$ $\quad dx = 5\sec\theta\tan\theta\,d\theta$

θ RANGE : $\cancel{-\pi/2 \le \theta < \pi \quad \pi}$ $0 \le \theta < \pi/2$

[illegible] $\le \theta < \pi$

$x = 5 \qquad \theta = 0$

$x = 10 \qquad \theta = \pi/3$

So $$\int_5^{10} \frac{1}{x\sqrt{\cancel{25-x^2}\ x^2-25}}\,dx =$$

$$\int \qquad\qquad d\theta =$$

$$=$$

$$\int \frac{dt}{\sqrt{t^2-6t+13}}$$

FIRST: COMPLETE THE SQUARE:

$$t^2-6t+13 =$$

8|

SUBSTITUTE:

$u =$ $du =$

$$\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{du}{\sqrt{}}$$

NOW TRIG. SUBSTITUTE:

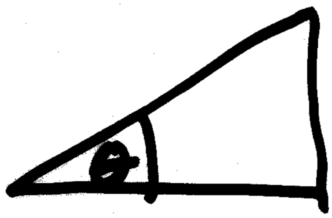
$u =$ $du =$

$$\int \frac{du}{\sqrt{u^2+4}} = \int \qquad d\theta$$

$$= \int \qquad d\theta$$

91

=



$\tan\theta =$ $=$

$\sec\theta =$

$=$

So

$$\int \frac{dt}{\sqrt{t^2 - 6t + 13}} = \qquad + C$$