Approximate Integration

Sometimes it is hard to find an anti derivative of a function. In fact, sometimes it is impossible to write one down in "Closed form".

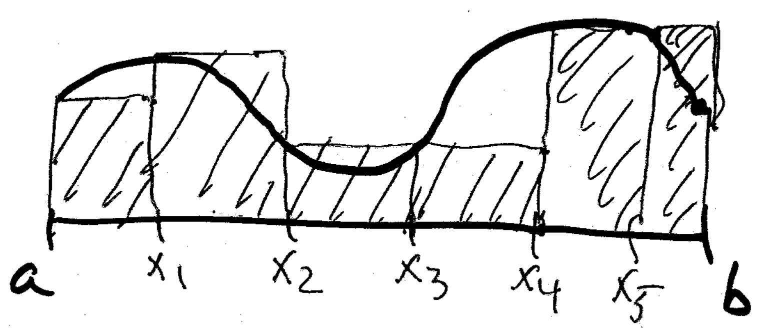
<u>Ex</u>: $\int e^{-x^2}\,dx$ exists, but there is no easy expression for it.

Also sometimes you want $\left[\int_a^b f(x)\,dx\right]$ but don't know $f(x)$ precisely, but know some values of $f(x)$,

<u>Ex</u>: Experimental data

In these cases what one can do is to approximate the integral, and we introduce some tools for this.

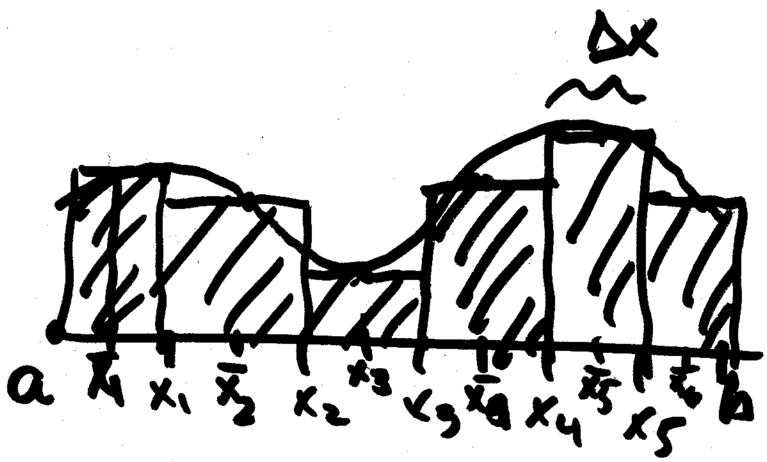
Recall $\int_a^b f(x)\,dx$ is a limit of areas of rectangles, summed together (possibly with signs)



For example this, above is the left endpoint sum for a certain choice of intermediate points

$$L_n(f) = \sum^{n} \Delta x \, f(x_{i-1})$$

One approximation is the <u>Midpoint</u> <u>sum</u> which we have already used:



$$\Delta x = \frac{b-a}{n}, \qquad \bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$$

$$\int_a^b f(x)\,dx \approx M_n = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots \right.$$

$$\left. \cdots + f(\bar{x}_n) \right]$$

$$x_0 = a, \quad x_n = b$$

$$x_i = a + i\left(\frac{b-a}{n}\right)$$

TRAPEZOIDAL RULE

Another approach is to average the Right hand sum (R_n) and the Left hand sum (L_n)

$$R_n = \sum_{i=1}^{n} \Delta x f(x_i) =$$

$$\Delta x \left(f(x_1) + f(x_2) + \cdots + f(x_n) \right)$$

$$L_n = \Delta x \left(f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right)$$

[Recall $a = x_0$, $b = x_n$, $x_i = a + \left(\frac{b-a}{n}\right) \cdot i$]

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left(R_n + L_n \right)$$

$$= \frac{\Delta x}{2} \Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \Big]$$

We call this T_n or $T_n(f)$ and it is called the Trapezoidal Approximation

Why? (Why is it called that?)

Draw the line segment from $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$

Then, instead of rectangles we have trapezoids. The area of a trapezoid is:

$\frac{1}{2} h\left(\frac{b_1+b_2}{2}\right)$, where h is the "height" b_1, b_2 the length of the bases.

In our case $h = \Delta x$

$b_1 = f(x_i)$, $b_2 = f(x_{i+1})$

Add the areas of these trapezoids $\left(\frac{\Delta x}{2}(f(x_i) + f(x_{i+1})\right)$ and we get

T_n.

Use the midpoint and trapezoidal rule to approximate

$\int_0^{\pi} \sin x \, dx$ using $n=4$

[of course in this case we can compute exactly, but the point is to find out how good these methods are]

$x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = \frac{3\pi}{4}$,

$x_4 = \pi$.

$\Delta x = \frac{\pi}{4} = \left(\frac{\pi - 0}{4}\right)$.

$\bar{x}_1 = \pi/8$, $\bar{x}_2 = \frac{3\pi}{8}$, $\bar{x}_3 = \frac{5\pi}{8}$, $\bar{x}_4 = \frac{7\pi}{8}$

$$M_4 = \frac{\pi}{4}\left[\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin(7\pi/8)\right]$$

$$\approx \frac{\pi}{4}(.383 + .924 + .924 + .383)$$

$$\approx 2.053$$

$$T_4 = \frac{\left(\frac{\pi}{4}\right)}{2}\left(\sin(0) + 2\sin(\pi/4) + 2\sin(\pi/2) + 2\sin(\tfrac{3\pi}{4}) + \sin(\pi)\right)$$

$$= \frac{\pi}{8}\left(0 + 2\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 2\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{\pi}{8}(2 + 2\sqrt{2}) \approx 1.896$$

Since $\int_0^{\pi} \sin x \, dx = 2$, these both seem reasonable.

M_4 is closer than T_4

But T_4 was easier to compute.

E_T, E_M (ERROR BOUNDS) CAN BE FOUND IN THE TEXT.

IF $|f(x)| \leq K$ on $[a, b]$

$$|E_T| \leq \frac{K(b-a)}{12n^2}$$ while

$$|E_M| \leq \frac{K(b-a)}{24n^2}$$

Apply this to $f(x) = \sin x$; $n = 4$

$|\sin x| \leq 1$ on $[0, \pi]$ so

$$|E_T| \leq \frac{\pi}{12 \cdot 16} \approx .016$$

$$|E_M| \leq \frac{\pi}{24 \cdot 16} \approx .008$$

#

How many terms n must we take so that

M_n approximates $\int_0^\pi \sin x \, dx$

within $.00001 = 10^{-5}$

How about T_n?

$|E_T| \le \frac{\pi}{12 \cdot n^2}$ want this smaller than 10^{-5}

$$\frac{\pi}{12n^2} \le 10^{-5}$$

$$n^2 \geq \frac{\pi}{12 \cdot 10^{-5}} = \frac{\pi 10^5}{12}$$

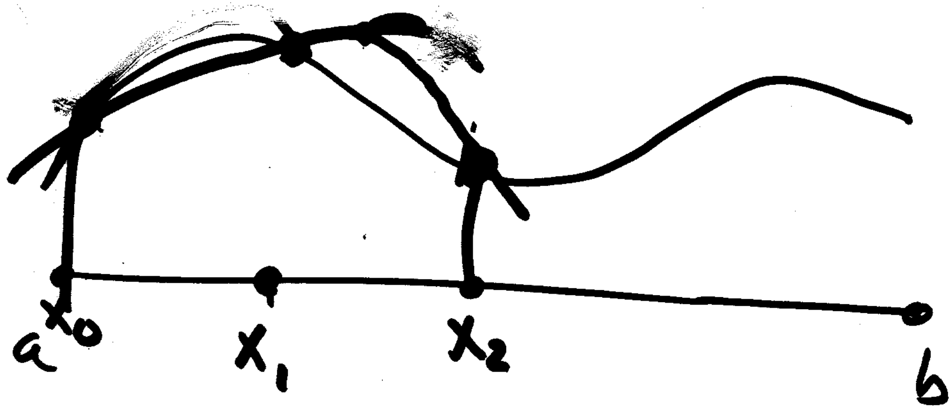
$$n > \sqrt{\frac{\pi 10^5}{12}} \approx 161$$

$$|E_m| \le 10^{-5} \Rightarrow \frac{\pi}{24n^2} < 10^{-5}$$

$$\Rightarrow n^2 > \frac{\pi 10^5}{24} \Rightarrow n > 114$$

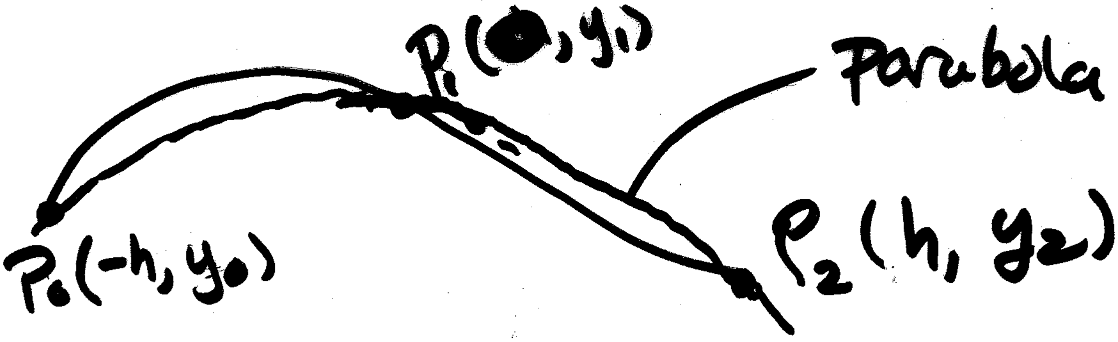
Simpson's Rule:

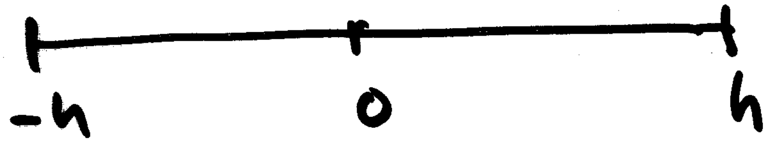
Instead of using straight lines, rectangles, try to use parabolas to approximate the integral.



Find a parabola that passes through those 3 pts

Simplest case $x_1 = 0$, $x_0 = -h$, $x_1 = h$





Want $Ax^2 + Bx + C$ to pass through $P_0(-h, y_0)$, $P_1(0, y_1)$, $P(h, y_2)$

Note: $\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_0^h (Ax^2 + C)\,dx$

$= 2\left[A\frac{x^3}{3} + Cx\right]_0^h = 2\left(A\frac{h^3}{3} + Ch\right)$

$= \frac{h}{3}\left[2A + 6C\right]$

R

Since $y = Ax^2 + Bx + C$ passes through the 3 points:

$$y_0 = A(-h)^2 + B(-h) + C$$
$$= Ah^2 - Bh + C$$

$$y_1 = A(0)^2 + B(0) + C = C$$

$$y_2 = Ah^2 + Bh + C$$

To combine these to get $2Ah^2 + 6C$ (as in the integral) we take

$$y_0 + 4y_1 + y_2$$

So without computing A, B, C we find the area of the approximating parabola:

$$\frac{h}{3}(y_0 + 4y_1 + y_2) \quad , \quad h = \Delta x$$

Similarly, the parabola passing through the points (x_2, y_2), (x_3, y_3) and (x_4, y_4) can have area:

$(y_i = f(x_i))$

$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$

Continue this approximation and we get:

$$\int_a^b f(x)\,dx \approx$$

$$\approx \frac{\overset{\Delta x}{\cancel{h}}}{3}(y_0 + 4y_1 + y_2) + \frac{\overset{\Delta x}{\cancel{h}}}{3}(y_2 + 4y_3 + y_4) +$$

$$\cdots + \frac{\overset{\Delta x}{\cancel{h}}}{3}(y_{n-2} + 4y_{n-1} + y_n) \quad (*)$$

<u>Note:</u> n must be <u>even</u>

$$\underbrace{(*) = \frac{\overset{\Delta x}{\cancel{h}}}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]}_{\text{Simpson's Rule.}}$$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx S_n =$$

$$\frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

where: $y_i = f(x_i)$

$x_i = \left(\frac{b-a}{n}\right)i + a$, $h = \frac{b-a}{n}$

<u>Ex:</u> $\int_0^{\pi} \sin x \, dx$: Approximate with

Simpson's rule, and $n = 4$;

$x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$

$y_0 = 0$ $y_1 = \sqrt{2}/2$ $y_2 = 1$ $y_3 = \sqrt{2}/2$, $y_4 = 0$

$$S_4 = \frac{(\pi/4)}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$$

$$= \frac{\pi}{12}\left[4 \cdot \frac{\sqrt{2}}{2} + 2 \cdot 1 + 4\frac{\sqrt{2}}{2}\right] = \frac{\pi}{12}\left[2 + 4\sqrt{2}\right]$$

≈ 2.0045

Simpson's Rule Error: E_S

$$|E_S| \le \frac{K(b-a)}{180n^4}$$

So if we use this rule to approximate $\int_0^\pi \sin x\,dx$ with $n=8$ we know we will at least be as close as

$$\frac{\pi}{180\cdot 8^4} \approx .000004$$

Recall, for this accuracy, M_n with $n = 114$ or T_n with $n = 161$.