SEQUENCES

A sequence is a list of numbers in a definite order

$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

Each number in the sequence is called a "term" and is referred to by its subscript, i.e.

a_1 is the "first term" of the sequence, a_2 is the "second term", etc., so a_{1734} is the "1,734th term" of the sequence.

We have several notations to indicate we are thinking of a sequence:

- $\{a_1, a_2, a_3, \ldots\}$ or $\{a_n\}$, or $\{a_n\}_{n=1}^{\infty}$ all mean the same thing

Examples: One way to define a sequence is to give a formula for the n^{th}-term.

(i) $\left\{\frac{n^2}{n^2+1}\right\}_{n=1}^{\infty}$, or $a_n = \frac{n^2}{n^2+1}$,

or $\left\{\frac{1}{2}, \frac{4}{5}, \frac{9}{10}, \ldots, \frac{n^2}{n^2+1} \ldots\right\}$

(ii) $\left\{\frac{(-1)^n \sqrt{n+1}}{2^{n+1}}\right\}_{n=1}^{\infty}$, or $a_n = \frac{(-1)^n \sqrt{n+1}}{2^{n+1}}$,

or $\left\{\frac{-\sqrt{2}}{4}, \frac{\sqrt{3}}{8}, \frac{-\sqrt{4}}{9}, \ldots, \frac{(-1)^n \sqrt{n+1}}{2^{n+1}} \ldots\right\}$

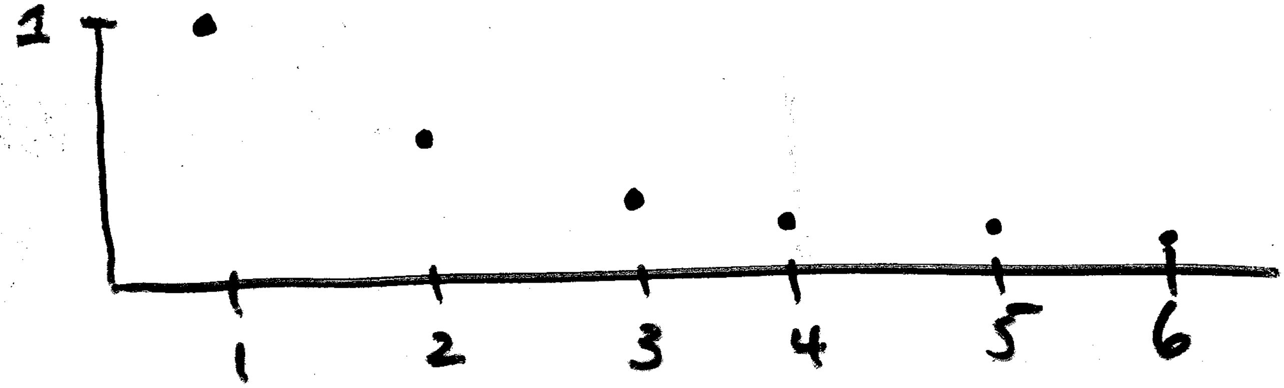
(iii) $\left\{\sin\left(\frac{n\pi}{3}\right)\right\}_{n=0}^{\infty}$, or, $a_n = \sin\left(\frac{n\pi}{3}\right), n \geq 0$

or, $\left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, +\frac{\sqrt{3}}{2}, \ldots\right\}$

<u>Note:</u> We can think of a sequence as a <u>function</u> on the <u>integers</u>.

So we <u>can</u> <u>graph</u> the sequence:

<u>Ex:</u> $a_n = \frac{1}{n}$, $n \geq 1$



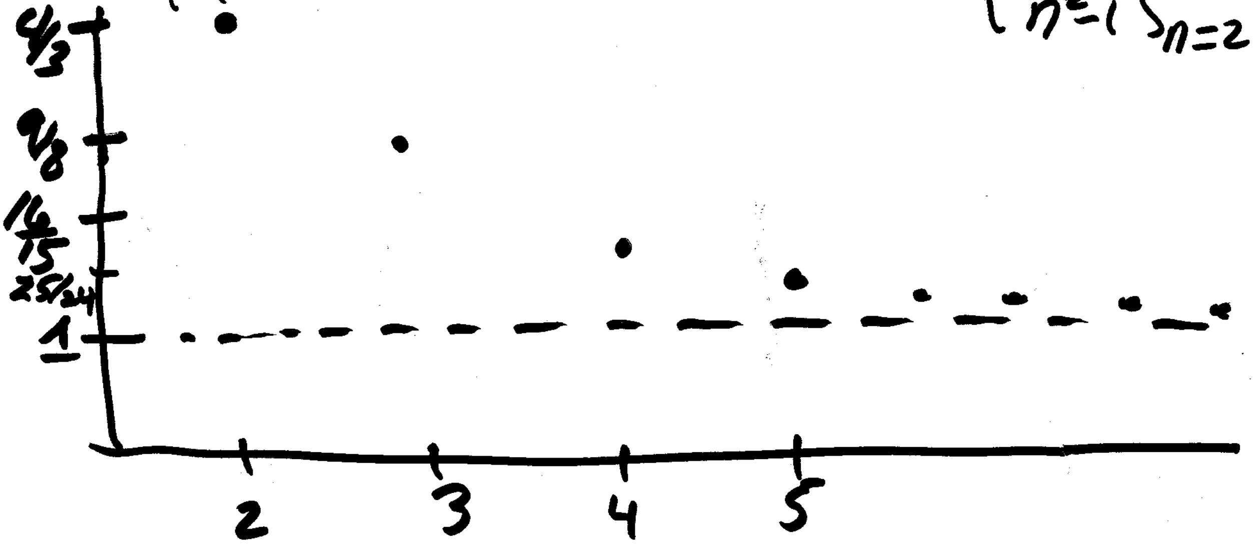
Another way to define a sequence is <u>recursively</u>, that is give a way to compute the next term from the one you know. <u>Ex:</u> Fibonacci Sequence:

$f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$

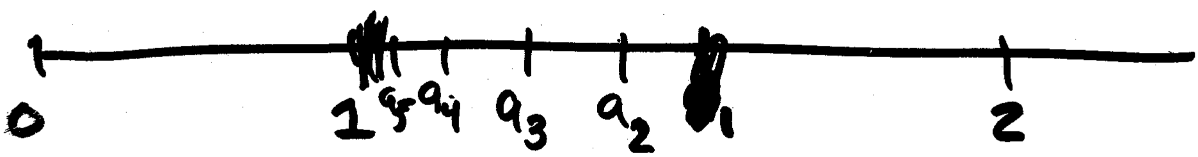
for $n \geq 2$

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 45 \ldots\}$

We can talk about the limit of a sequence: For example $\left\{\frac{n^2}{n^2-1}\right\}_{n=2}^{\infty}$



Or graph the sequence along the line



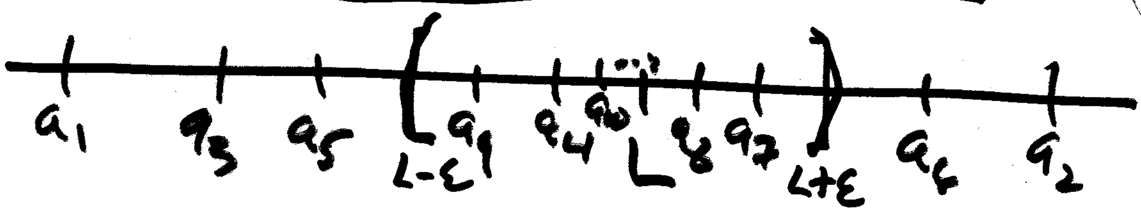
Note that $\frac{n^2}{n^2-1} - 1 = \frac{n^2-(n^2-1)}{n^2-1} = \frac{1}{n^2-1}$

This says we can make a_n as close to 1 as we like, provided we let n get large enough, (and for all subsequent n as well).

So we say $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2-1} = 1$.

<u>Definition:</u> The sequence $\{a_n\}$ has <u>limit L</u> if for any $\varepsilon > 0$ there is an $N = N_\varepsilon$ so that if $n > N$ $|a_n - L| < \varepsilon$.

$\boxed{\text{We write } \lim\limits_{n\to\infty} a_n = L,}$ or $\boxed{a_n \longrightarrow L}$ as $n \to \infty$



This means eventually, for any $\varepsilon > 0$, the sequence has all its terms in the interval $(L-\varepsilon, L+\varepsilon)$.

Note how this is similar to the definition $\lim_{x\to\infty} f(x) = L$

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\lim_{n\to\infty} a_n = L$.

Ex: $\lim_{x\to\infty} \frac{1}{x} = 0$, so if $a_n = \frac{1}{n}$, $n \geq 1$, then $\lim_{n\to\infty} a_n = 0$

IF $\lim_{n\to\infty} a_n = L$, i.e. if the limit exists, then, we say the sequence converges (or is convergent). Otherwise the sequence diverges.

We say $\lim_{n\to\infty} a_n = \infty$ if for any $M > 0$ there is an N so that $a_n > M$ for all $n > N$.

Note IF $\lim_{n\to\infty} a_n = \infty$ then the sequence is DIVERGENT!

There are limit laws for sequences similar to those for limits of functions:

e.g. $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

PROVIDED BOTH LIMITS ON THE RIGHT HAND SIDE OF THE EQUAL SIGN EXIST !!!!

Note: These laws are given on Page 696

SQUEEZE THM APPLIES TO SEQUENCES

IF $a_n \le b_n \le c_n$ for all n **and**

if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then

$\lim_{n\to\infty} b_n = L$.

IMPORTANT FACT:

IF $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

Ex: Find $\lim_{n\to\infty} \frac{n^2}{n^2-1}$

$$\lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \left(\frac{1}{1-1/n^2}\right)$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 - \lim_{n\to\infty} 1/n^2} = \frac{1}{1+0} = 1$$

$$\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} = ?$$

$$\frac{n}{1+\sqrt{n}} = \frac{n}{1+n^{1/2}} = \frac{n^{1/2}\cdot n^{1/2}}{n^{1/2}\cdot\left(\frac{1}{n^{1/2}}+1\right)}$$

$$\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\frac{1}{\sqrt{n}}+1}$$

$$= \frac{\lim\limits_{n\to\infty} \sqrt{n}}{\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}+1}$$ but $\lim\limits_{n\to\infty} \sqrt{n}$ <u>D.N.E</u>

$$\left(\lim_{n\to\infty} \sqrt{n} = \infty\right)$$

So $\frac{n}{1+\sqrt{n}}$ is <u>divergent</u>

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

$$n! = 1\cdot 2\cdot 3\cdot\cdots\cdot n$$

$$n^n = \underbrace{n\cdot n\cdot \cdots\cdot n}_{n\text{-factors}}$$

$$a_1 = \frac{1!}{1^1} = \frac{1}{1} = 1$$

$$a_2 = \frac{2!}{2^2} = \frac{1\cdot 2}{2\cdot 2} = 1/2$$

$$a_3 = \frac{3!}{3^3} = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3} = 2/9$$

$$\vdots$$

$$a_n = \frac{1\cdot 2\cdot 3\cdot \cdots\cdot n}{n\cdot n\cdot n\cdot \cdots\cdot n} =$$

$$\frac{1}{n}\left(\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdots\frac{n-1}{n}\right) \le \frac{1}{n}\cdot(1\cdot 1\cdot\cdots 1)$$

Note $\quad 0 \le a_n \le \frac{1}{n}$

Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} 1/n$,

$\lim_{n\to\infty} \frac{n!}{n^n} = 0$ by <u>Squeeze Theorem</u>.

How about $a_n = r^n$ for some fixe r.

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } |r| > 1 \text{ or DNE} \\ 1 & \text{if } r = 1 \\ 0 & \text{if } |r| < 1 \\ \text{Does not exist if } r = -1. \end{cases}$$

<u>Ex</u>: $\left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \to \infty$.

Since $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ can be made as small as possible.

Note ~~Next~~ $a_n = (-1)^n$,

$\{1, -1, 1, -1, 1, -1, \ldots\}$ has no limit.

A sequence is increasing if $a_n \geq a_{n-1}$ for all n. A sequence is decreasing if $a_n \leq a_{n-1}$ for all n.

Ex: $\frac{1}{n}$ is a decreasing sequence

$\frac{n}{n+1}$ is an increasing sequence

since

$$\frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - (n+1)(n-1)}{(n+1)n}$$

$$= \frac{n^2 - (n^2-1)}{(n+1)n} = \frac{1}{(n+1)n} > 0 \quad \text{i.e.}$$

$$\frac{n}{n+1} > \frac{n-1}{n}.$$

A sequence is bounded above if there is some M with $a_n \leq M$ for all n.

Ex:

Since $\frac{n}{n+1} < 1$,

$\{\frac{n}{n+1}\}$ is bounded above.

A sequence is <u>bounded below</u> if there is some M with

$M \leq a_n$ for all n;

Ex: $\frac{1}{n}$ is bounded below since

$\frac{1}{n} > 0$ for all $n \geq 1$.

Every bounded monotonic sequence converges:

<u>BOUNDED</u> = BOUNDED ABOVE AND BELOW

<u>MONOTONIC</u> = INCREASING/DECREASING

Use differentiation: For increasing/decreasing, limits (L'Hopital's rule)

Ex: $a_n = \dfrac{n}{2^n}$, $n \geq 2$

Consider $f(x) = \dfrac{x}{2^x}$.

$f'(x) = \dfrac{2^x \cdot 1 - (x \ln 2) 2^x}{2^{2x}}$ (quotient rule)

$= \dfrac{1 - x\ln 2}{2^x} < 0$ if $x \ln 2 > 1$ i.e. $x > \dfrac{1}{\ln 2}$, $n \geq 2$.

So f is decreasing $\Longrightarrow$ a_n is decreasing.

a_n is bounded, below by 0, above by 1, so a_n is convergent.

Ex: $\frac{\ln(n^2)}{n} = a_n$

Is $\{a_n\}$ convergent?

$\lim_{n\to\infty} a_n = ?$

If $f(x) = \frac{\ln(x^2)}{x}$, to find $\lim_{x\to\infty} f(x)$ we use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{\ln(x^2)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^2}\cdot 2x}{1}$$

$$= \lim_{x\to\infty} \frac{2}{x} = 0.$$

So $\lim_{n\to\infty} \frac{\ln(n^2)}{n} = 0.$

SERIES

If $\{a_n\}$ is a sequence we can ask if there is any way to make sense of

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

Intuition: You shouldn't be able to sum up infinitely many terms.

On the other hand, remember improper integrals, like

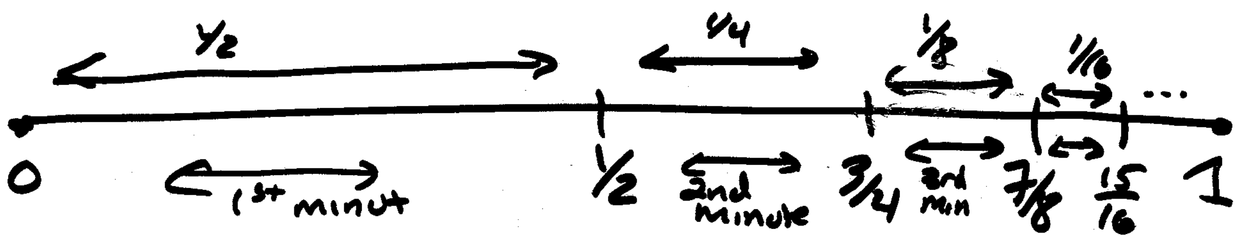
$\int_1^{\infty} \frac{1}{x^2}\,dx$. We thought this couldn't make sense either.

We will say more about the connection between improper integrals and series next class

Consider the "sum"

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

Think of a street of length 1 mile. Every minute you walk halfway from the current position to the end.



So it seems that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

$a_i = \frac{1}{2^i}$ is our sequence.

Let $S_n = a_1 + a_2 + \dots + a_n$

This is a <u>partial</u> <u>sum</u> of the sequence

3.

S_n - n^{th} partial sum.

For our sequence

$S_1 = \frac{1}{2}$

$S_2 = 3/4$

$S_3 = 7/8$

$S_4 = 15/16$

$S_5 = 31/32$

$S_6 = \frac{63}{64}$

$\vdots$

$S_n = \frac{2^n - 1}{2^n}$

So the <u>sequence</u> <u>of</u> <u>partial</u> <u>sums</u> $\{S_n\}$ converges to 1.

<u>Definition:</u> If $\{a_n\}$ is a sequence write $\sum a_n$ or $\sum_{n=1}^{\infty} a_n$ for the <u>series</u> $a_1 + a_2 + \cdots + a_n + \cdots$ (as a <u>formal</u> <u>expression</u>)

Def: Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its n^{th} partial sum,

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

If the sequence $\{s_n\}$ is convergent, and if $\lim_{n\to\infty} s_n = s$ we say $\sum a_n$ is convergent and write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or}$$

$$\sum_{n=1}^{\infty} a_n = s$$

If $\lim_{n\to\infty} s_n$ does not exist, then the series is called divergent

<u>Example:</u> Let r be a fixed ~~positive~~ number, and $\underline{a}$ is a constant.

Consider the series

$$a + ar + ar^2 + ar^3 + \dots + ar^n + \dots$$

Assume $r \neq 1$, since $a + a + \dots + a$ will diverge.

Consider the partial sum S_n:

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} \quad (1)$$

Note

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n \quad (2)$$

Subtract (1) from (2):

$$(r-1)S_n = ar^n - a = a(r^n - 1)$$

So

$$S_n = \frac{a(r^n-1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)}$$

So $\{S_n\}$ converges <u>if and only if</u> $\lim_{n\to\infty} r^n$ exists

Recall (Sec. 11.1), $r^n \longrightarrow 0$ if $|r|<1$

and $\{r^n\}$ is divergent otherwise (for $r \neq 1$)

Thus:

The geometric series:

$$(a \neq 0) \quad \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^n + \cdots$$

is convergent if $\underline{\underline{|r|<1}}$ and

then
$$\sum_{n=1}^{\infty} ar^{n-1} = a\left(\frac{1}{1-r}\right) = \frac{a}{1-r}.$$

IF $|r| \geq 1$ The geometric series diverges.

<u>EX:</u> $a=1$, $r=\frac{1}{2}$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$$

Consider

$$4 - \frac{8}{5} + \frac{16}{25} - \frac{32}{125} + \frac{64}{625} - \cdots$$

This is a <u>geometric series</u> with

$a = 4$ (first term) and

$$r = \frac{\left(-\frac{8}{5}\right)}{4} = -2/5$$

<u>Note</u>: $\frac{\left(\frac{16}{25}\right)}{\left(-8/5\right)} = -2/5 = \frac{\left(-\frac{32}{125}\right)}{\left(16/25\right)} = \cdots$

Our series is $\sum_{n=1}^{\infty} 4 \cdot \left(\frac{-2}{5}\right)^{n-1}$

so we get

$$\sum_{n=1}^{\infty} 4\left(-2/5\right)^{n-1} = \frac{4}{1-\left(-2/5\right)} = \frac{4}{\left(7/5\right)} = \frac{20}{7}$$

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TELESCOPING SERIES:

$$\sum_{n=1}^{\infty} \frac{1}{2n(n+1)}$$

By Partial fractions:

$$\frac{1}{2n(n+1)} = \frac{1}{2n} - \frac{1}{2(n+1)}$$

So our partial sums are

$$S_n = \frac{1}{2\cdot 1\cdot 2} + \frac{1}{2\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2n(n+1)} =$$

$$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2(n-1)} - \frac{1}{2n}\right) + \left(\frac{1}{2n} - \frac{1}{2(n+1)}\right)$$

$$= \frac{1}{2} - \frac{1}{2(n+1)}$$

9.

Since:

$S_n \longrightarrow \frac{1}{2}$ as $n \longrightarrow \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2}.$$

Harmonic Series:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges:

$= 1 + \frac{1}{2} + \frac{1}{3} + \dots$

Why?

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$

Since $\frac{1}{3} > \frac{1}{4}$, $\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$, so

$S_4 > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

Since $\frac{1}{5} > \frac{1}{6} > \frac{1}{7} > \frac{1}{8}$, so

$S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$

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$$S_{16} = 1 + \frac{1}{2} + \cdots + \frac{1}{8} + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 4/2$$

For any n;

$$S_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{n}{2}$$

So $\lim_{n\to\infty} s_n = \infty$, does not converge.

IMPORTANT FACT

IF $\sum_{n=1}^{\infty} a_n$ is convergent

then $\lim_{n\to\infty} a_n = 0$

NOTE REVERSING "IF" AND "THEN" DOES **NOT** WORK.

EX: $\frac{1}{n} \longrightarrow 0$

but $\sum \frac{1}{n}$ **diverges**

Divergence Test If $\lim_{n\to\infty} a_n$ does not exist, or $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$

$$a_n = \frac{(n+1)^2}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n}$$

$$= \frac{n^2}{n^2}\left(\frac{1+2/n+1/n^2}{1+2/n}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1+2/n+\frac{1}{n^2}}{1+2/n}\right)$$

$$= \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{2}{n} + \lim_{n\to\infty}\frac{1}{n^2}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2/n}$$

$$= \frac{1+0+0}{1+0} = 1$$

So $\sum \frac{(n+1)^2}{n(n+2)}$ <u>diverges</u>

Rules for Infinite Series:

Suppose $\sum a_n$ and $\sum b_n$ are both convergent.

Then, for any constant c,

$\sum c\,a_n$, $\sum (a_n + b_n)$, and

$\sum (a_n - b_n)$ are all convergent.

Furthermore:

$$\sum_{n=1}^{\infty} c\,a_n = c \sum_{n=1}^{\infty} a_n ,$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

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Use these rules to find

$$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)} - \frac{4}{3^n}\right)$$

We saw $\boxed{\sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2}}$

so $\sum \frac{1}{n(n+1)} = \sum 2\cdot\left(\frac{1}{2n(n+1)}\right)$

$= 2(1/2) = 1$

Also $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \cdots$

$= (-1+1) + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

$= -1 + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$

$= -1 + \frac{1}{1-1/3} = -1 + \frac{3}{2} = 1/2$

So $\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)} - \frac{4}{3^n}\right) =$

$\sum_{n=}^{\infty} \frac{1}{n(n+1)} - 4\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2} - 4\left(\frac{1}{2}\right)$

$= -3/2$