Integral Test

Determining whether or not a series converges is difficult. Then, even if you can show $\sum a_n$ converges, it might be very difficult to find the sum. One way to decide whether $\sum a_n$ converges is known as the integral test.

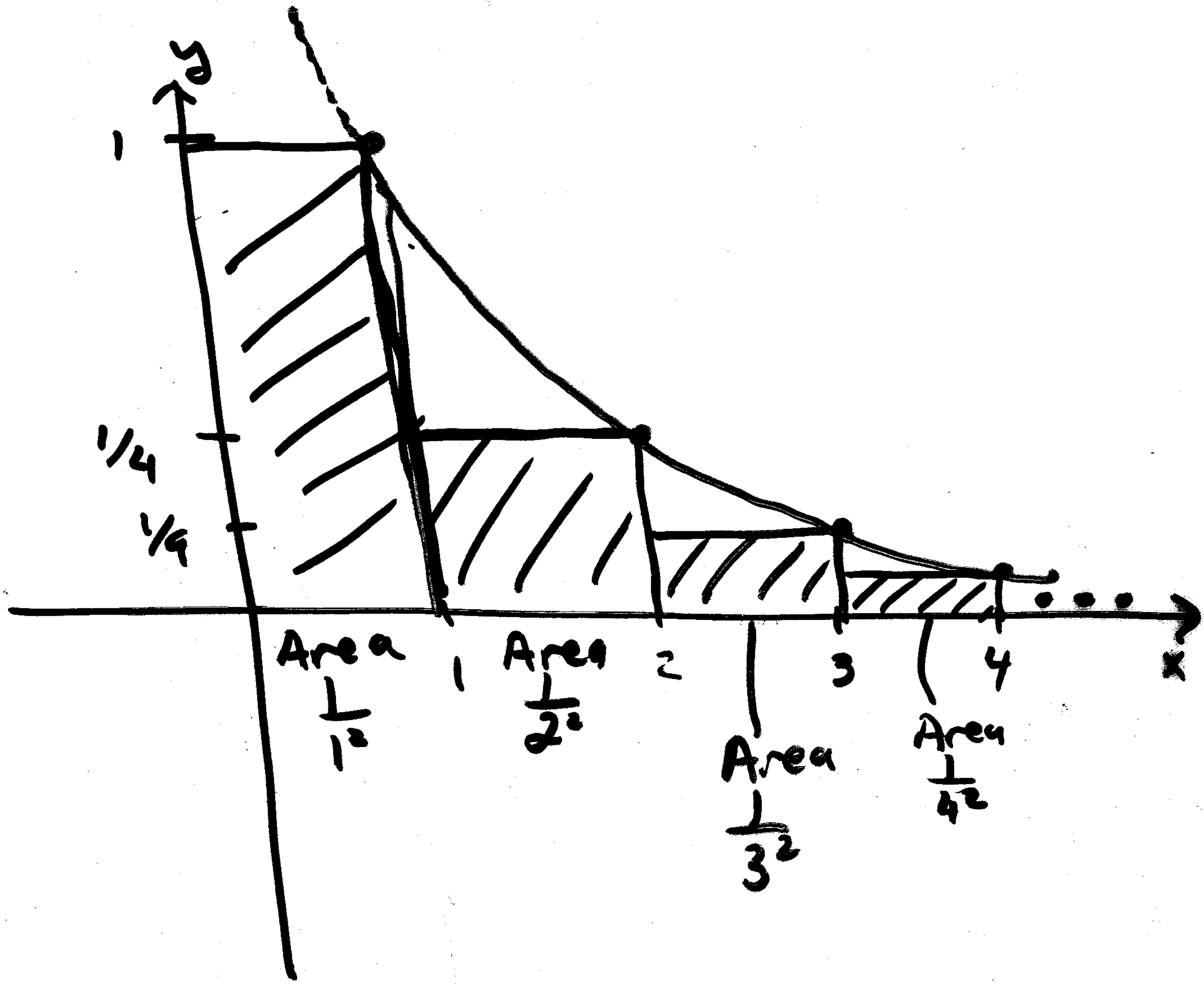
Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} =$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

No simple formula for S_n;

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} = ?$$

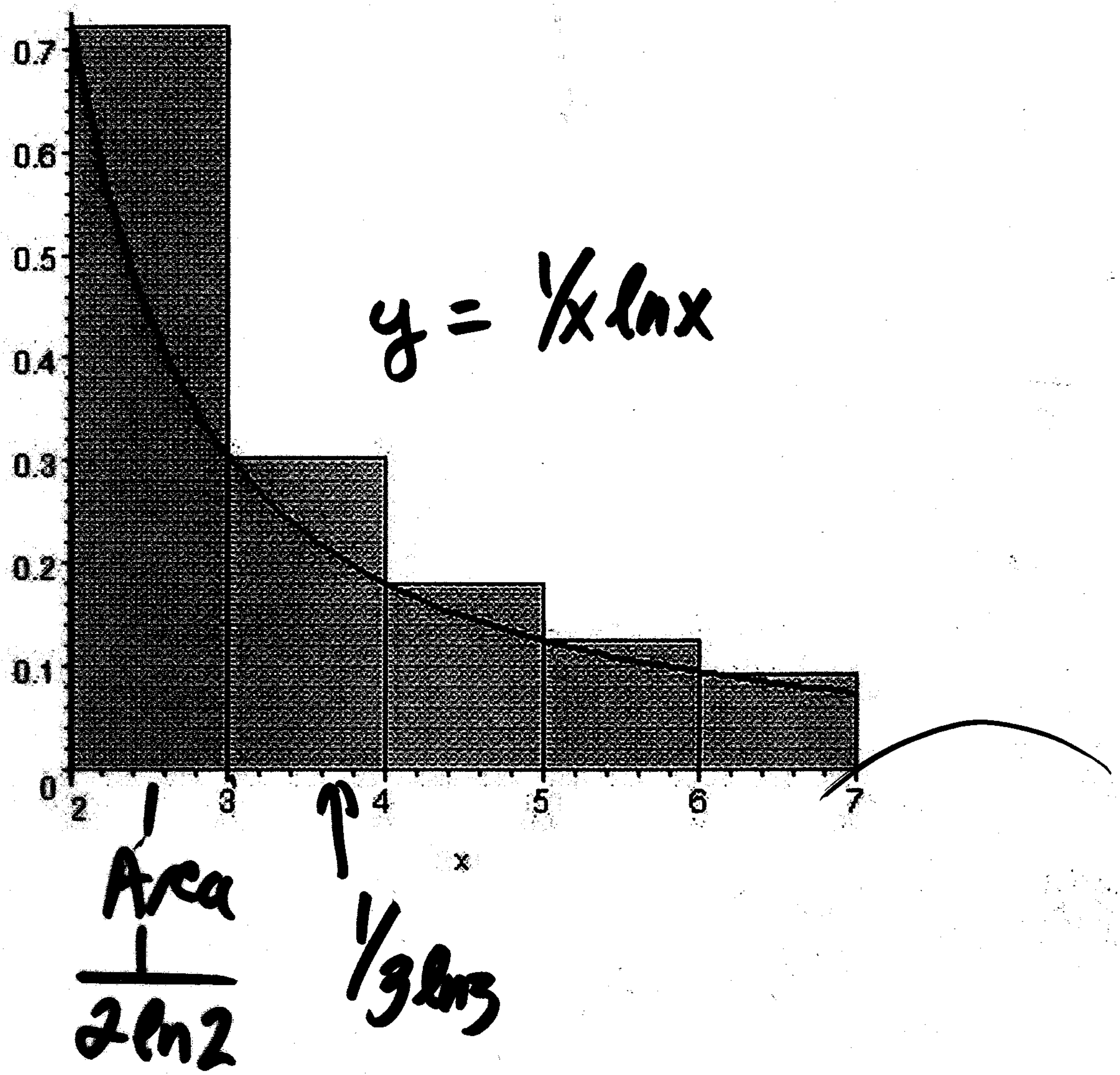
Consider $f(x) = 1/x^2$



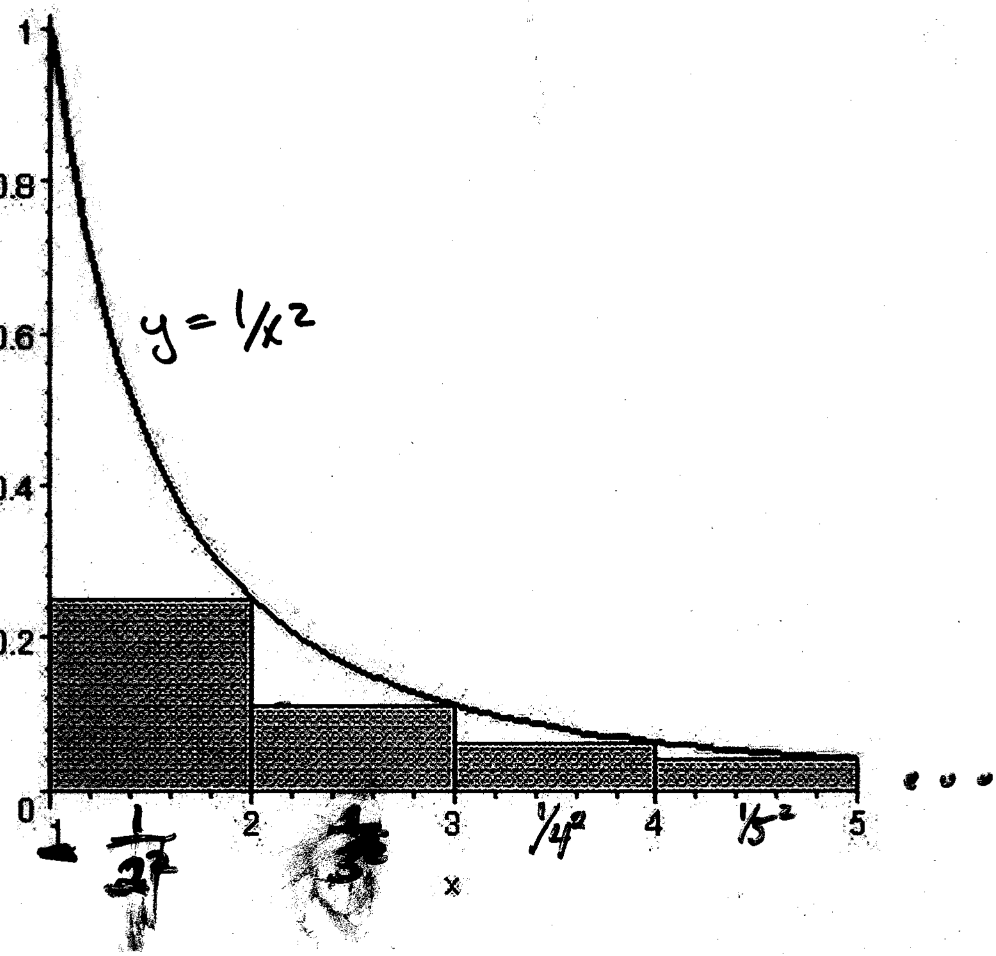
The area of rectangles:

$$\frac{1}{1^2} + 1/2^2 + \cdots + \frac{1}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Ignore first term; Then all rectangles lie below $y = \frac{1}{x^2}$ for $1 \le x < \infty$



0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2
3
4
5
6
7
x
y = 1/x ln x
Area
1/2ln2
1/3ln3



$y = 1/x^2$
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
x
$\frac{1}{2^2}$
$1/4^2$
$1/5^2$

(3) We know $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$ (Ch. 7, Sec. 8)

So $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots < 1$

Therefore $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots < 2$

and in fact converges.

On the other hand consider

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Now using left hand sums;

$$\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \cdots + \frac{1}{n \ln n} + \cdots$$

is greater than the area under

$f(x) = \frac{1}{x \ln x}$, $\infty > x \geq 2$

$$\lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} \ln(\ln|x|)\Big|_2^t =$$

$$\lim [\ln(\ln t) - \ln\ln(2)] = \infty$$

So since $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is greater than this integral $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

<u>Theorem</u>: (Integral Test)

Suppose $f(x)$ is a <u>POSITIVE</u> <u>CONTINUOUS</u> <u>DECREASING</u> function on $[1, \infty)$ and $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges <u>if and only if</u> $\int_1^{\infty} f(x)\,dx$ converges : i.e.

(i) If $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges

(ii) If $\int_1^{\infty} f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

(5) As in our second example we may use this test for $\sum_{n=2}^{\infty} a_n$, etc.

Ex: $\sum_{n=5}^{\infty} \frac{1}{(n-2)^3}$ converges

only if $\int_5^{\infty} \frac{1}{(x-2)^3}\,dx$ converges.

<u>Ex:</u> Is $\sum_{n=1}^{\infty} e^{-n}$ convergent?

Let $f(x) = e^{-x}$.

(i) $f(x) > 0$ for all x. (ii) $f(x)$ is continuous on $[1, \infty)$. (iii) $f(x)$ is decreasing, i.e. if $a > b$, $f(a) \leq f(b)$.

So apply the integral test:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} (-e^{-x})\Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{1}{e} - \frac{1}{e^t}\right) = \frac{1}{e}, \text{ so}$$

the series converges.

(6) NOTE: The integral test only tells you whether or not the series converges, but not what its value is; e.g.

we showed (using integral test)

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and

$\sum_{n=2}^{\infty} \frac{1}{n^2} \le 1$ so $\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$.

It is not the case that

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \int_2^{\infty} \frac{1}{x^2}\,dx$$

In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, but we won't be able to show that.

P-series:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$. When is it convergent ??

Recall $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p \le 1$

⑦ So $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/5}}$

diverges, since $\frac{2}{5} \leq 1$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

REMARK: NO ONE KNOWS THE EXACT SUM OF THIS SERIES... $$

Ex: $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{1}{4n-1}.$$

Let $f(x) = \frac{1}{4x-1}$. Note: $\displaystyle\int_1^{\infty} \frac{1}{4x-1}\,dx =$

$$\lim_{t\to\infty} \int_1^t \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \frac{1}{4}\left(\ln(4x-1)\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{4} \ln\left(\frac{4t-1}{3}\right) = \infty$$

So the series diverges

ESTIMATING SUMS.

Suppose we know $\sum_{n=1}^{\infty} a_n$ converges.

Then $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists.

If $S = \sum_{n=1}^{\infty} a_n$, then

$$S = \lim_{n} \sum_{i=1}^{n} a_i$$, so we

can estimate S to any degree of accuracy by choosing an N large enough to guarantee

$|S - S_N|$ is small enough

This is what "limit exists" means. We can, in fact estimate the error.

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This is what "limit exists" means. We can, infact estimate the error.

Suppose $a_n = f(n)$.

Using the integral test

we can say that

$R_n = s - s_n$ satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

<u>Note:</u> $R_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + \cdots$

<u>Ex:</u> $\sum_{n=1}^{\infty} \frac{1}{n^4}$. How good an estimate is $\sum_{n=1}^{10} \frac{1}{n^4}$?

We know it is at worst:

$$\int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_{10}^{t} x^{-4}\,dx$$

$$= \lim_{t\to\infty} -\frac{1}{3} x^{-3} \Big|_{10}^{t} = \lim_{t\to\infty} \frac{1}{3}\left(10^{-3} - t^{-3}\right) = \frac{1}{3\cdot 10^3}$$

Thus the sum

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} =$$

$1.0820365 84\ldots$ is accurate to at least 3 decimal places.

The error is also no better than

$$\int_{11}^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3 \cdot 11^3} = \frac{1}{3993}$$

$$= .00025043826 70 \ldots$$

How many terms until $S_n = \sum_{k=1}^{n} \frac{1}{k^4}$ is an approximation to $\sum_{k=1}^{\infty} \frac{1}{k^4}$ with accuracy 10^{-6}?

Need $\frac{1}{3 \cdot n^3} < \frac{1}{100000}$

$n^3 > \frac{100000}{3}$, $n > \left(\frac{100000}{3}\right)^{1/3} \approx 32$

Comparison tests:

Does $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converge?

Compare with $\sum 1/n^2$.

$\frac{1}{n^2+3} < \frac{1}{n^2}$, so for any n,

The partial sums s_n satisfy

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2+3} < \sum_{k=1}^{n} \frac{1}{k^2} = S_n$$

We saw $S_n < 2$, (using integral test)

So $s_n < S_n < 2$. So $\{s_n\}$ is a bounded increasing sequence $\Longrightarrow \lim_{n\to\infty} s_n$ exists (See Section 11.1). So

$\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges.

Theorem: (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are two series with **positive** terms, and such that:

$$a_n \leq b_n \quad \text{forall } n.$$

(i) If $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges

③ Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{n^3} = ?$

Since $\displaystyle\frac{n^2+4}{n^3} > \frac{n^2}{n^3} = \frac{1}{n}$,

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{n^3}$ diverges (using (ii))

LIMIT COMPARISON TEST

Thm: Suppose $\sum a_n$ and $\sum b_n$ are two series with positive terms. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, a finite non-zero number (i.e. $0 < c < \infty$) then either both series converge or both series diverge. (i.e. if $\sum a_n$ converges so does $\sum b_n$, and if $\sum a_n$ diverges so does $\sum b_n$).

4) Our last example:

$\sum 1/n$ diverges and

$$\lim_{n\to\infty} \left(\frac{n^2+4}{n^3}\right) \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n^3+4n}{n^3}$$

$$= \lim_{n\to\infty} \left(1 + \frac{4}{n^2}\right) = 1, \text{ so}$$

$\sum_{n=1}^{\infty} \frac{n^2+4}{n^3}$ also diverges

(this is another proof of this fact).

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{n+1}{n\,2^n}$.

We know $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, and

$$\lim_{n\to\infty} \frac{\left(\frac{n+1}{n\,2^n}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \text{ so } \sum_{n=1}^{\infty} \frac{n+1}{n2^n}$$

converges.

<u>NOTE:</u> THE CONDITION, $0 < c < \infty$ IS CRUCIAL

<u>Ex:</u> Consider $a_n = \frac{1}{n^2}$, $b_n = 1/n$, $c_n = 1/n^3$.

We know $\sum a_n$ and $\sum c_n$ converge, while $\sum b_n$ diverges.

Note now that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

and $\lim\limits_{n\to\infty} \frac{a_n}{c_n} = \lim\limits_{n\to\infty} n = \infty$

<u>So these limits give no conclusion</u>

Does $\sum_{n=1}^{\infty} \frac{7+3\cos\left(\frac{n\pi}{256}\right)}{n^3}$

Converge or diverge?

—

Use the comparison test:

Since $-3 \le 3\cos\left(\frac{n\pi}{256}\right) \le 3$,

$$4 \le 7+3\cos\left(\frac{n\pi}{256}\right) \le 10$$

Thus, $\frac{7+3\cos\left(\frac{n\pi}{256}\right)}{n^3} \le \frac{10}{n^3}$.

Since $\sum_{n=1}^{\infty} \frac{10}{n^3}$ converges, so

does $\sum_{n=1}^{\infty} \frac{7+3\cos(n\pi/256)}{n^3}$

Ex:

$$\sum \frac{2n+5}{n^2+3n+2}$$

Note the degree of the denominator is one greater than that of the numerator. So compare (limit) with $1/n$

$$a_n = \frac{2n+5}{n^2+3n+2} \qquad b_n = 1/n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+5}{n^2+3n+2}\right)}{(1/n)} =$$

$$\lim_{n\to\infty} \frac{2n^2+5n}{n^2+3n+2} = \lim_{n\to\infty} \frac{2+\frac{5}{n}}{1+\frac{3}{n}+2/n^2}$$

$= 2.$ $\boxed{\text{So } \sum \frac{2n+5}{n^2+3n+2} \text{ diverges}}$

Ex:

$$\sum_{n=1}^{\infty} \frac{1+e^n}{e^{2n}} .$$

Note $\frac{1+e^n}{e^{2n}} > \frac{1}{e^n}$, so we <u>cannot</u> use the comparison test.

On the other hand $1+e^n \approx e^n$

So we expect $\frac{1+e^n}{e^{2n}} \approx 1/e^n$

Use limit comparison with $\frac{1}{e^n}$.

$$\lim_{n\to\infty} \frac{\left(\frac{1+e^n}{e^{2n}}\right)}{(1/e^n)} = \lim_{n\to\infty} \frac{e^n + e^{2n}}{e^{2n}} =$$

$$\lim_{n\to\infty} \frac{e^{2n}}{e^{2n}}\left(1+\frac{1}{e^n}\right) = 1.$$

Is $\sum_{n=1}^{\infty} \frac{n!}{(2n)!\,5^n}$ convergent?

Notice: $\frac{n!}{(2n)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) \cdots 2n}$

$= \frac{1}{(n+1)(n+2)\cdots(2n)} < 1$

So $a_n = \frac{n!}{(2n)!\,5^n} = \left(\frac{n!}{(2n)!}\right)\frac{1}{5^n}$

$< \frac{1}{5^n} = b_n$. Use Comparison test.

$\sum_{n=1}^{\infty} \frac{1}{5^n}$ Converges, so $\sum \frac{n!}{(2n)!\,5^n}$

Converges as well.

Alternating series

What if not all terms of

$\sum a_n$ are positive?

We say a series is <u>alternating</u>

if the terms are alternately

negative and positive: e.g.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$-\frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{3n+1}\right)$$

note if a term is positre the next term is negative, and VICE VERSA.

So an alternating series

$\sum a_n$ is given by either:

$a_n = (-1)^n b_n$, or $a_n = (-1)^{n+1} b_n$ ②

with $b_n > 0$

<u>Alternating Series Test:</u>

Suppose the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots,$$

with $b_n > 0$, satisfies

$$\boxed{\begin{array}{ll} \text{(i)} & b_{n+1} \le b_n \text{ for all } n \\ \text{(ii)} & \lim_{n\to\infty} b_n = 0 \end{array}}$$

THEN THE SERIES <u>CONVERGES</u>

EXAMPLE:

The alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$$

Converges since

(i) $\frac{1}{n+1} \leq \frac{1}{n}$

(ii) $\lim_{n \to \infty} \frac{1}{n} = 0.$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n^2+1}{4n^2+1}$$

$\lim_{n \to \infty} \frac{2n^2+1}{4n^2+1} = 1/2 \neq 0$, so

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{(-1)^n(2n^2+1)}{4n^2+1}$,

does not exist so by the Divergence test the series diverges

How about

$$\sum \frac{(-1)^n \ln(n^2)}{n^2}$$

It is true that (use L'Hopital's rule)

$$\lim_{n\to\infty} \frac{\ln(n^2)}{n^2} = \lim_{n\to\infty} \frac{(2n/n^2)}{2n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

But is $b_{n+1} \le b_n$?

Consider $f(x) = \dfrac{\ln(x^2)}{x^2}$

$$f'(x) = \frac{x^2\left[2x/x^2\right] - 2x\ln(x^2)}{x^4}$$

$$= \frac{2 - 2\ln(x^2)}{x^3} = 2 - \ln(x^4)$$

As soon as $x^4 > e^2$ i.e. $x^2 > e$

$f'(x) < 0$, so f is decreasing

$\sqrt{e} \approx 1.5$ 5

at least on $[2, \infty)$.

Thus $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n^2)}{n^2}$

Converges.

ESTIMATING ALT. SERIES

$S = \displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$

Again we let $S_n = \cancel{\displaystyle\sum_{k=1}^{n}}$

$= \displaystyle\sum_{k=1}^{n} (-1)^n b_n$. $R_n = S - S_n$

Estimate: Suppose $b_n \geq b_{n+1}$

for all n and $\displaystyle\lim_{n\to\infty} b_n = 0$.

Then $|R_n| = |S - S_n| \leq b_{n+1}$

Ex: $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Then $|S - S_n| \leq \dfrac{1}{n+1}$

Estimate $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n \cdot n!}$ to

3 decimal places:

WE HAVE $\lim_{n\to\infty} \frac{1}{2^n \cdot n!} = 0$,

and $\frac{1}{2^{n+1}(n+1)!} < \frac{1}{2^n \cdot n!}$,

So in fact the series converges

and $|R_n| \leq \frac{1}{2^{n+1}(n+1)!}$

If we want $|R_n| \leq \frac{1}{1000}$
we have to have

$$\frac{1}{1000} \geq \frac{1}{2^{n+1} \cdot (n+1)!} \Rightarrow 2^{n+1}(n+1)! \geq 1000$$

Note $2^{10} = 1024$, so $n+1 < 10$;

$$2^4 \cdot 4! = 16 \cdot 24 = 384$$

$$2^5 \cdot 5! = \underbrace{32 \cdot 120}_{3840} > 3000$$

So $n+1 = 5$, $n = 4$.

$$\sum_{n=1}^{4} \frac{(-1)^n}{2^n n!} = -\frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384}$$

$$= \frac{-151}{384} \approx -.3932\ldots$$

Approximates $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2^n \cdot n!}\right)$

to 3 decimal places

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$$ Converges

or not?

n	$\sin(n\frac{\pi}{2})$
1	$\sin(\pi/2) = 1$
2	$\sin(\pi) = 0$
3	$\sin(3\pi/2) = -1$
4	$\sin(2\pi) = 0$
5	$\sin(5\frac{\pi}{2}) = 1$
	$\vdots$

$\Rightarrow$

$$\sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n!} = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!}$$

$$\lim_{n\to\infty} \frac{1}{(2n-1)!} = 0,$$

$$\frac{1}{(2n-1)!} > \frac{1}{(2n+1)!}.$$

How many terms until

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$$ is approximated,

To within $1/10{,}000$

Need $\frac{1}{(2n+1)!} < \frac{1}{10{,}000}$

or $(2n+1)! > 10{,}000$

$7! = 5040$.

$8! = 40320$

So $\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$ approximates

$$\sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n!}$$ to within $1/10{,}000$

(In fact since

$9! = 362{,}880$ this

approximates within $\frac{1}{362{,}880} =$

$.0000027755\ldots$