Taylor Polynomials

Suppose $f(x)$ equals its Taylor expansion at a;

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Define the k^{th} Taylor Polynomial approximation to $f(x)$ at a by

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)$$

$$T_0(x) = f(a)$$

$$T_1(x) = f(a) + f'(a)(x-a)$$

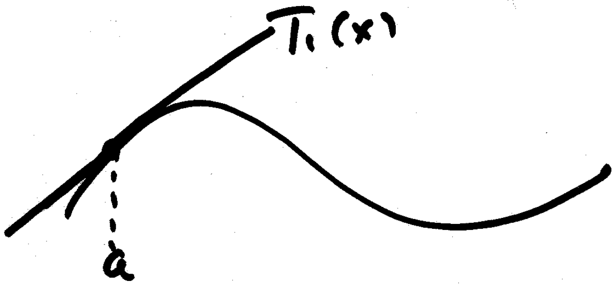
$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$\vdots$$

<u>Note:</u> $T_1(x) =$

$$f(a) + f'(a)(x-a)$$

is the tangent line approximation.



In general $T_n^{(k)}(a) = f^{(k)}(a)$ for

$0 \leq k \leq n$.

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Ex:

$f(x) = \cos x$

Mc Claurin Series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$T_0(x) = 1 = T_1(x)$

$T_2(x) = 1 - \frac{x^2}{2} = T_3(x)$

$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} = T_5(x)$

$\vdots$

How good does This approximate $f(x)$??

Taylor's Inequality:

If there is a constant M so that

$$|f^{(n+1)}(x)| \leq M \quad \text{for all}$$

$|x-a| \leq d$, then the Taylor polynomial $T_n(x)$ satisfies

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},$$

for $|x-a| \leq d$.

We call $R_n(x) = f(x) - T_n(x)$ the <u>Taylor</u> <u>remainder</u>

Ex: $f(x) = e^{2x}$

Recall $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$

Substituting $u = 2x$, we get

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

Find the 3rd degree Talyor polynomial at $a=0$.

$a=0$: $e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3$

$= T_3(x)$.

How good is this approximation?
Say on $|x| < 2$.

$f^{(4)}(x) = 2^4 e^{2x} = 16e^{2x}$.

On $-2 \le x \le 2$, $|f^{(4)}(x)| \le 16e^4$

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This is our M in this case.

$$|R_3(x)| = |e^{2x} - T_3(x)| \le \frac{16e^4 |x|^4}{4!}$$

$$= 2\frac{e^4|x|^4}{3}$$ (Not very good for $|x|$ larger than 1.)

What about 3rd Taylor polynomial at $a = \ln 2$?

$$f(\ln 2) = e^{2\ln 2} = e^{\ln(2^2)} = 4$$

$$f'(\ln 2) = 2e^{2\ln 2} = 8$$

$$f''(\ln 2) = 4e^{2\ln 2} = 16$$

$$f'''(\ln 2) = 32$$

$$T_3(x) = 4 + 8(x - \ln 2) + 8(x - \ln 2)^2 + \frac{16}{3}(x - \ln 2)^3$$

6/ The error is:

$$M \frac{|x - \ln 2|^4}{4!}.$$

Since $|f^{(4)}(x)| = |16e^{2x}|$ on $-2 \le x \le 2$, we still have

$M = 16e^4$

$$|R_3(x)| \le \frac{2e^4 |x - \ln 2|^4}{3}$$

Ex: Taylor polynomial of $f(x) = \cos x$ at $x = 2\pi/3$

$f(2\pi/3) = \cos(2\pi/3) = -\frac{1}{2}$

$f'(2\pi/3) = -\sin(2\pi/3) = -\frac{\sqrt{3}}{2}$

$f''(2\pi/3) = -\cos(2\pi/3) = 1/2$

$f'''(2\pi/3) = \sin(2\pi/3) = \sqrt{3}/2$

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$$\cos x = -\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{2\pi}{3}\right) + \frac{1}{2\cdot 2!}\left(x-\frac{2\pi}{3}\right)^2$$

$$+ \frac{\sqrt{3}}{2\cdot 3!}\left(x-\frac{2\pi}{3}\right)^3 - \frac{1}{2\cdot 4!}\left(x-\frac{2\pi}{3}\right)^4 - \frac{\sqrt{3}}{2\cdot 5!}\left(x-\frac{2\pi}{3}\right)^5$$

$$+ \cdots$$

$$T_0 = -\frac{1}{2}$$

$$T_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - 2\pi/3\right)$$

$$T_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - 2\pi/3\right) + \frac{1}{4}\left(x-\frac{2\pi}{3}\right)^2$$

$$T_3(x) = -\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{2\pi}{3}\right) + \frac{1}{4}\left(x-\frac{2\pi}{3}\right)^2 +$$

$$\frac{\sqrt{3}}{12}\left(x-\frac{2\pi}{3}\right)^3.$$

So, since $\left| f^{(4)}(x) \right| \le 1$

we have

$$\left| T_3(x) - \cos(x) \right| \le \frac{1}{4!} \left| x - 2\pi/3 \right|^4$$

Ex: $f(x) = x^{-2}$; $a = 1$

2nd degree Taylor polynomial

$0.9 \le x \le 1.1$

$f(1) = 1$

$f'(1) = -2(1)^{-3} = -2$

$f''(1) = 6(1)^{-4} = 6$

$f'''(x) = -24x^{-5}$

$T_2(x) = 1 - 2(x-1) + 6(x-1)^2$

$\left| f'''(x) \right| = 24 \left| \frac{1}{x^5} \right| \le 24 \left| \frac{1}{(9/10)^5} \right|$

$= 24 \left(\frac{10^5}{9^5} \right) \; (= M)$

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$$R_2(x) \le 24 \cdot \left(\frac{10}{9}\right)^5 \cdot \frac{|x-1|^3}{3!}$$

$$\le 24 \cdot \frac{10^5}{9^5} \cdot \frac{10^{-3}}{3!}$$

Applications

The resistivity ρ of a conductor wire is measured in ohm-meters (Ω-m). Resistance depends on temperature as:

$$\rho(t) = \rho_{20}\, e^{\alpha(t-20)}$$

t is temp in °C. The values α and ρ_{20} (depend on the metal) are computed experimentally. Find 1st and 2nd Talyor polynomials (at $t=20$)

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$$\rho(20) = \rho_{20} e^{0} = \rho_{20}$$

$$\rho'(20) = \alpha\rho_{20} e^{0} = \alpha\rho_{20}$$

$$\rho''(20) = \alpha^2\rho_{20} e^{0} = \alpha^2\rho_{20}$$

$$T_1(t) = \rho_{20} + \alpha\rho_{20}(t-20)$$

$$T_2(t) = \rho_{20} + \alpha\rho_{20}(t-20) + \frac{\alpha^2\rho_{20}}{2}(t-20)^2$$

For Copper: $\alpha = .0039/°C$

$\rho_{20} = 1.7 \times 10^{-8}\ \Omega\text{-m}$

Let's Graph T_1, T_2 for

$-250° \leq t \leq 1000°C$.