Functions as
POWER SERIES

Recall a power series is a series:

$$\sum_{n=0}^{\infty} c_n (x-a)^n \; ; \; \boxed{x \text{ is a variable}}$$

In at least one case we know it represents a function:

(1) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

FOR $|x| < 1$

Other functions can also be given as a power series.

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<u>Ex:</u> $\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$

Replace x by $-x^3$ in (1)

$$\frac{1}{1-(-x^3)} = 1+(-x)^3 + (-x^3)^2 + (-x^3)^3 + \cdots + (-x^3)^n + \cdots$$

$$= 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n},$$

Converges for $|x^3| < 1$, i.e. $|x| < 1$.

<u>Ex:</u> $\frac{x}{1+2x} = x\left(\frac{1}{1-(-2x)}\right)$

(2) $= x\left(1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots (-2x)^n + \cdots\right) =$

$$x\left(\sum_{n=0}^{\infty} (-2x)^n\right) = x \sum_{n=0}^{\infty} (-2)^n x^n =$$

$$\sum_{n=0}^{\infty} (-2)^n x^{n+1} \; ;$$

<u>Note:</u> to get equation (2)

we substituted $\underline{-2x}$ for $\underline{x}$ in (1).

Since (1) converges for $|x| < 1$,

(2) converges for $|2x| < 1$, i.e.

for $|x| < 1/2$.

$$\frac{1}{9+x^2} = \frac{1}{9}\left(\frac{1}{1+(x/3)^2} \right)$$

$$= \frac{1}{9}\left(\frac{1}{1-(-(x/3)^2)} \right)$$

Replace x by $(-(x/3)^2)$ in equation (1)

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$$\frac{1}{9+x^2} = \frac{1}{9}\left(\frac{1}{1-\left(-\left(\frac{x}{3}\right)^2\right)}\right) =$$

$$\frac{1}{9}\sum_{n=0}^{\infty}\left[\left(-\frac{x}{3}\right)^2\right]^n =$$

$$\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^{2n}x^{2n} =$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^{n}x^{2n} =$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^{n+1}x^{2n}$$

Converges for $\left|\left(-\frac{x}{3}\right)^2\right| < 1$

$|x^2| < 9$ or

$|x| < 3$.

Ex: $\dfrac{1}{x-5} = \dfrac{1}{(x-4)-1} = \dfrac{-1}{1-(x-4)}$

$-\sum_{n=0}^{\infty} (x-4)^n$, converges for

$|x-4| < 1$ i.e.,

$-1 < x-4 < 1$ or

$3 < x < 5$

Term by Term Differentiation and integration.

Suppose: $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

is convergent for $|x-a| < R$.

Then $f'(x) = \sum_{n=0}^{\infty} c_n \cdot n (x-a)^{n-1}$

$= c_1 + 2c_2(x-a) + \cdots + n(x-a)^{n-1} + \cdots$

This series also has radius of convergence R.

Also,

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $|x-a| < R$.

That is, differentiation and integration of power series is accomplished term by term

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$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \quad |x|<1$$

Differentiate both sides

$$\frac{-3x^2}{(1+x^3)^2} = \sum_{n=1}^{\infty} (-1)^n 3n\, x^{3n-1}$$

(note the beginning index).

EX: $\int \frac{1}{1-x}\,dx = -\ln(1-x) + C$

$|x|<1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

$$\ln(1-x) = C - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \left(C = 0 \text{ by plugging in } x=0 \right)$$

Ex.

Since $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

and

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n, \quad = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

for $|x|<1$,

So

$$\arctan x = \int \frac{1}{1+x^2}\,dx =$$

$$C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \text{ for } |x|<1$$

Can make further substitutions:

Ex: $\arctan x^2 =$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1},$$

$|x| < 1$

$$\ln(1+3x) = \ln(1-(-3x))$$

$$= -\sum_{n=1}^{\infty} (-3x)^n/n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n x^n}{n}$$

Converges for $|-3x| < 1$, or

$|x| < 1/3$.