POWER SERIES

A **POWER SERIES** IS

A SERIES OF THE FORM

$$(1)\quad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

WITH X A **VARIABLE**.

THE CONSTANTS $\{c_n\}$ ARE CALLED THE **COEFFICIENTS** OF THE SERIES

FOR EACH X WE CAN ASK IF (1) CONVERGES OR DIVERGES

WE ALREADY KNOW MANY POWER SERIES:

(2) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$$= \frac{1}{1-x} \quad -1 < x < 1$$

and DIVERGES OTHERWISE

(3) <u>ANY</u> <u>POLYNOMIAL</u> <u>IS</u> A POWER SERIES

$c_0 + c_1 x + \cdots + c_n x^n + 0 x^{n+1} + \cdots$
$+ 0 x^m + \cdots$

i.e. all but finite number of c_i are zero

A POWER SERIES <u>CENTERED AT</u> <u>$x = a$</u>;

$$\sum_{n=0}^{\infty} c_n (x-a)^n =$$

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

ALSO CALLED A POWER SERIES <u>ABOUT</u> <u>a</u>

<u>EXAMPLE</u>: FIND ALL X FOR WHICH $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ CONVERGES

USE RATIO TEST:

$a_n = x^n/n!$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n}$$

$$= \frac{|x|}{n+1} \quad ; \quad \text{so}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1, \text{ for any } x \text{ thus}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{CONVERGES}$$

FOR ALL x.

HOW ABOUT

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n} \quad ?$$

Again we USE THE RATIO TEST:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 |x|^{n+1}}{10^{n+1}} \cdot \frac{10^n}{n^2 \cdot |x|^n}$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot \frac{|x|}{10}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{|x|}{10}$$

$$= \frac{|x|}{10}.$$

SO THE SERIES CONVERGES ABSOLUTELY IF $\frac{|x|}{10} < 1$

i.e $-10 < x < 10$,

and diverges for $|x| > 10$.

FOR $x = \pm 10$ WE MUST CHECK EXPLICITLY.

That is because the ratio test is inconclusive when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

$$x = 10; \quad \sum_{n=1}^{\infty} \frac{n^2 \cdot 10^n}{10^n} = \sum_{n=1}^{\infty} n^2$$

diverges by the divergence test: $\lim_{n\to\infty} n^2 = \infty$.

$$x = -10, \quad \sum_{n=1}^{\infty} \frac{n^2(-10)^n}{10^n} = \sum_{n=1}^{\infty} (-1)^n n^2$$

also diverges, since $\lim_{n\to\infty} (-1)^n n^2$ doesn't exis

<u>Ex:</u> $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n}$. For what x does the series converge?

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$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-4|^{n+1}}{2(n+1)} \cdot \frac{2n}{|x-4|^n} \Bigg\} \text{Ratio Test}$$

$$= |x-4| \cdot \frac{n}{n+1}$$

So $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-4|$

FOR $|x-4| < 1$, $\sum\limits_{n=\emptyset}^{\infty} \frac{(x-4)^n}{2n}$

CONVERGES ABSOLUTELY

$$-1 < x-4 < 1$$
$$3 < x < 5$$

FOR $x > 5$ or $x < 3$

$(|x-4| > 1)$ THE SERIES

DIVERGES.

$x = 3$; $\sum\limits_{n=0}^{\infty} \frac{(3-4)^n}{2n}$

$= \frac{1}{2} \sum\limits_{n=\emptyset}^{\infty} \frac{(-1)^n}{n}$ CONVERGES (CONDITIONALLY)

$x = 5$; $\sum\limits_{n=1}^{\infty} \frac{(5-4)^n}{2n} = \frac{1}{2} \sum\limits_{n=1}^{\infty} \frac{1}{n}$

DIVERGES

THEOREM:

Given a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are 3 possibilities:

(i) The series converges only when $x=a$

(ii) The series converges for all x

(iii) There is a positive number R so that the series converges* if $|x-a| < R$ and diverges if $|x-a| > R$

*abs.

WE CALL R the <u>radius of convergence</u>

$$a-R < x < a+R$$

In (ii) we say the radius of convergence is ∞; and in (i) the radius of convergence is 0.

Interval of convergence:

(iii)
$$\left.\begin{array}{l}(a-R,\ a+R)\\(a-R,\ a+R]\\ [a-R,\ a+R)\\ [a-R,\ a+R]\end{array}\right\}\ \text{possibilities depends on convergence at endpoints}$$

(i) Interval $\{a\}$

(ii) Interval $(-\infty, \infty)$

EXAMPLES:

Geometric series:

$\sum_{n=0}^{\infty} x^n$; $R=1$, $(-1,1)$

Any polynomial

$c_0 + c_1 x + \cdots + c_n x^n$, $R=\infty$, $(-\infty,\infty)$

$\sum \frac{x^n}{n!}$; $R=\infty$, $(-\infty,\infty)$

$\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$; $R=10$; $(-10,10)$

$\sum \frac{(x-4)^n}{2n}$ $R=1$; $[3,5)$