Rational Functions & Partial fractions (Review)

- $\int \frac{P(x)}{Q(x)}\,dx$ with $P(x)$, $Q(x)$

Polynomials.

1. If degree $P(x) \geq$ degree $Q(x)$,

divide $P(x)$ by $Q(x)$, (long division),
write $P(x) = P_1(x)\,Q(x) + R(x)$,
with degree $R(x) <$ degree $P(x)$.

Now
$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)}$$

Assume now $\deg P(x) < \deg Q(x)$

2. Factor $Q(x)$:

$Q(x) = Q_1(x)\,Q_2(x)\cdots Q_k(x)$

where each $Q_i(x)$ is either

linear : $(Q_i(x) = (ax+b))$ or

irreducible quadratic

$(Q_i(x) = ax^2 + bx + c,\ b^2 - 4ac < 0)$.

3. Partial fractions: 4 cases

Case i: Each $Q_i(x)$ is linear and no repeated factors. To each factor

$$ax+b \longleftrightarrow \frac{A}{ax+b}$$ and

$\frac{P(x)}{Q(x)}$ is the sum:

Ex: $\displaystyle\int \frac{x^2+2}{x^2-5x+6}\,dx$

$$\frac{x^2+2}{x^2-5x+6} = \frac{x+2}{(x-3)(x-2)} =$$

$$\frac{A}{x-3} + \frac{B}{x-2}$$

Clear denominators

$$x+2 = A(x-2) + B(x-3) =$$
$$(A+B)x + (-2A-3B)$$

$$\Rightarrow \begin{cases} A+B=1 \\ -2A-3B=2 \end{cases}$$ (multiply 1st row by 2)

$$\Rightarrow \begin{cases} 2A+2B=2 \\ -2A-3B=2 \end{cases}$$ (add 1st row to 2nd)

$$\Rightarrow \begin{cases} 2A+2B=2 \\ -B=4 \end{cases}$$ (or $A+B=1$) $\Rightarrow$ $B=-4$

$A=5$

$$\int \frac{x^2+2}{x^2-5x+6}\,dx = \int \frac{5}{x-3}\,dx - \int \frac{4}{x-2}\,dx$$

Case (ii) Repeated linear factors:

to each $(ax+b)^m \longleftrightarrow$

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

then $\frac{P(x)}{Q(x)}$ is the sum:

EX: $\displaystyle\int \frac{2x+2}{(x-2)^2(x+4)}\,dx$

$$\frac{2x+2}{(x-2)^2(x+4)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x+4)}$$

Clear denominators:

$$2x+2 = A(x-2)(x+4) + B(x+4) + C(x-2)^2$$

$$= A(x^2+2x-8) + B(x+4) + C(x^2-4x+4)$$

$$= (A+C)x^2 + (2A+4B-4C)x + (-8A+4B+4C$$

$$\begin{cases} A+C = 0 \\ 2A+B-4C=2 \\ -8A+4B+4C=2 \end{cases} \Longrightarrow$$ (solve first for C or A and substitute)

$$\begin{cases} A=-C \\ 6A+B=2 \\ -12A+4B=2 \end{cases} \longrightarrow$$ (Replace 3rd row by its sum with twice 2nd row)

$$\begin{cases} A=-C \\ 6A+B=2 \\ 6B=6 \end{cases} \longrightarrow \begin{array}{l} B=1 \\ A=1/6 \\ C=-1/6 \end{array}$$

$$\int \frac{2x+2}{(x-2)^2(x+4)}\,dx = \int \frac{1}{6(x-2)}\,dx + \int \frac{1}{(x-2)^2}\,dx$$

$$+\int \frac{-1}{6(x+4)}\,dx$$

Case (iii) $\quad ax^2+bx+c \quad , \; b^2-4ac<0$

no repeated quad. factors. $\longleftrightarrow \quad \dfrac{Ax+B}{ax^2+bx+c}$

Ex: $\displaystyle\int \frac{t+4}{(t^2+1)(t-2)}\,dt$

Write

$$\frac{t+4}{(t^2+1)(t-2)} = \frac{A}{t-2} + \frac{Bt+C}{t^2+1}$$

$$\begin{aligned} t+4 &= A(t^2+1) + (Bt+C)(t-2) \\ 0t^2+1t+4 &= A(t^2+1) + Bt^2+(C-2B)t-2C \\ &= (A+B)t^2 + (C-2B)t + (A-2C) \end{aligned}$$

$$\begin{cases} A+B = 0 \\ -2B+C = 1 \\ A \quad -2C = 4 \end{cases} \xrightarrow{\text{similar to others}} \quad \begin{aligned} A &= 6/5 \\ B &= -6/5 \\ C &= -7/5 \end{aligned}$$

$$\int \frac{t+4}{(t^2+1)(t-2)}\,dt =$$

$$\frac{6}{5}\int \frac{1}{t-2}\,dt - \frac{1}{5}\int \frac{6t+7}{t^2+1}\,dt$$

$$= \frac{6}{5}\ln|t-2| - \frac{1}{5}\int \frac{6t}{t^2+1}\,dt - \frac{7}{5}\int \frac{1}{t^2+1}\,dt$$

$\downarrow$ subst. $u = t^2+1$

$$= \frac{6}{5}\ln|t-2| - \frac{3}{5}\ln|t^2+1| - \frac{7}{5}\tan^{-1}t + C$$

Note: "Cover up Method" for

(i) $X+2 = A(x-2) + B(x-3)$

at $x=2$: $4 = -B \Rightarrow B = -4$

at $x=3$: $5 = A$

(ii) $2x+2 = A(x-2)(x+4) + B(x+4) + C(x-2)^2$

at $x=2$: $6 = 6B \Rightarrow B=1$

$x=-4$: $-6 = 36C$, $C = -1/6$

Need to solve for A: $A+C=0$

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(iii) $t + 4 = A(t^2+1) + (Bt+C)(t-2)$

at $t = 2$:

$6 = 5A$, $A = \frac{6}{5}$

Still need to solve for B, C

Strategies for integration:

1. KNOW TABLE ON PG. 500

2. SIMPLIFY INTEGRAND
(EX: DIVIDE POLYNOMIAL OF SMALLER DEGREE INTO ONE OF LARGER DEGREE, OR REPLACE $\tan x$ BY $\frac{\sin x}{\cos x}$ (or VICE-VERSA) MIGHT HELP

Ex.

$$\int \frac{\cot\theta}{\csc^2\theta} = \int \frac{\cos\theta}{\sin\theta}\cdot \sin^2\theta \, d\theta$$

$$= \int \cos\theta \sin\theta \, d\theta$$

3. LOOK FOR SUBSTITUTION:

In (iii) above we had

$\boxed{\int \frac{6t}{t^2+1}\, dt}$, take $u = t^2+1$
$du = 2t\, dt$

4. LOOK FOR "KNOWN FORMS"

i.e. Things we have learned to deal with

(a) Trigonometric functions

$\sin^n x \cos^m x$ or

$\tan^n x \sec^m x$ or

$\cos(mx)\sin(nx)$ etc.

(b) Rational functions (covered)

(C) Integration by parts:

Look for expressions dealt with this way; $x^n \ln(\text{—})$, $x^n e^{(\;)}$,

$x^n \sin(\;)$, $x^n \cos(\;)$, $e^{(\;)}\cos(\;)$

etc.

LOOK FOR $f(x)$ you can take $f'(x)$ easily, and $g'(x)$ you can take

(d) Radicals

(i) $\sqrt{\pm x^2 \pm a^2}$ use trig substitution

(ii) $\sqrt[n]{ax+b}$ rationalize by using substitution $u = \sqrt[n]{ax+b}$

Ex. $\int \sin\sqrt{3t}\, dt$;

$u = \sqrt{3t} \Rightarrow u^2 = 3t$

$\Rightarrow 2u\,du = 3\,dt$

$\Rightarrow dt = \frac{2}{3} u\,du$

$$\int \sin\sqrt{3t}\, dt = \frac{2}{3}\int u \sin u\, du$$

Then use parts.

Manipulate the integrand:

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\sin x}\left(\frac{1-\sin x}{1-\sin x}\right)dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx - \int \frac{\sin x\,dx}{\cos^2 x}$$

use known facts and substitution.

Use a table of integrals:
or CAS (MATLAB/MAPLE)

Ex:

$$\int \frac{\sqrt{4-9x^2}}{x}\,dx$$

Line 32.

$$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

$a=2,\ u=3x\ ,\ x=\frac{u}{3};\quad dx=\frac{du}{3}$

$$\int \frac{\sqrt{4-9x^2}}{x}\,dx = \int \frac{\sqrt{a^2-u^2}}{\left(\frac{u}{3}\right)}\,\frac{du}{3}$$

$$= \sqrt{4-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

$$= \sqrt{4-9x^2} - 2\ln\left|\frac{2+\sqrt{4-9x^2}}{3x}\right| + C$$

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$$\int 6x \sin^{-1}(\pi x)\, dx$$

LINE 90:

$$\int u \sin^{-1} u \, du =$$

$$\frac{2u^2-1}{4} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$$

$$u = \pi x; \qquad x = \frac{u}{\pi} \qquad dx = \frac{du}{\pi}$$

$$\int 6x \sin^{-1}(\pi x)\, dx = 6\int \left(\frac{u}{\pi}\right) \sin^{-1}(u) \frac{du}{\pi}$$

$$= \frac{6}{\pi^2}\left[\frac{2u^2-1}{4} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C\right]$$

$$= \frac{6}{\pi^2}\left[\frac{2\pi^2x^2-1}{4} \sin^{-1}(\pi x) + \frac{\pi x\sqrt{1-\pi^2x^2}}{4}\right]$$