11.7 Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus, the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 □ $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

EXAMPLE 2 □ $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-se

STRATEGIES:

To determine whether a given series $\sum a_n$ converges, we have several tests.

p-series test:
$\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$.

Geometric Series test.

$\sum ar^n$ converges if $|r|<1$ and diverges if $|r| \geq 1$

We also can compare series which are similar to Geometric or p-series.
ESPECIALLY RATIONAL OR ALGEBRAIC SERIES

EXAMPLES:

RATIONAL EXPRESSIONS

$$\sum \frac{4n^3+3n-9}{7n^8-5n^2+2}.$$

Compare with $\frac{4n^3}{7n^8}$ or

just $\frac{1}{n^5}$. ($\sum 1/n^5$ converges)

$$\lim_{n\to\infty} \frac{\left(\frac{4n^3+3n-9}{7n^8-5n^2+2}\right)}{\left(\frac{1}{n^5}\right)} =$$

$$\lim_{n\to\infty} \frac{4n^8+3n^6-9n^5}{7n^8-5n^2+2} =$$

$$\lim_{n\to\infty} \frac{n^8}{n^8}\left(\frac{4+\frac{3}{n^2}-\frac{9}{n^3}}{7-\frac{5}{n^6}+\frac{2}{n^8}}\right)$$

$= 4/7$; Since $0 < \frac{4}{7} < \infty$, the series converges.

limit comparison test

ALGEBRAIC EXPRESSIONS

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^5+3n}}{2n^2-4}$$

Compare with

$$\frac{\sqrt{n^5}}{2n^2} = \frac{n^{1/2}}{2}$$

Limit comparison $\longrightarrow$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^5+3n}}{2n^2-4} \quad \text{diverges}$$

Also divergence test :

$$\lim_{n\to\infty} \frac{\sqrt{n^5+3n}}{2n^2-4} =$$

$$\lim_{n\to\infty} \frac{n^{5/2}\sqrt{1+\frac{3}{n^4}}}{n^2\,(2-4/n^2)}$$

$$= \lim_{n\to\infty} n^{1/2}\left(\frac{\sqrt{1+3/n^4}}{2-4/n^2}\right) = \infty \neq 0$$

ALTERNATING SERIES

$$\sum (-1)^n b_n \quad , \qquad b_n > 0$$

try to use alt. series test.

Ex: $$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

Consider $f(x) = \frac{\ln x}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\frac{1}{2\sqrt{x}}} \quad (*)$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

$$f'(x) = \frac{\sqrt{x}\left(\frac{1}{x}\right) - \frac{\ln x}{2\sqrt{x}}}{x}$$

$$= \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x\sqrt{x}} = \frac{2 - \ln x}{2x\sqrt{x}} < 0$$

for $\ln x > 2$, ($x > e^2$), so f is decreasing. THUS THE SERIES CONVERGES BY ALT. SERIES TEST

IF $a_n = (b_n)^n$
may want to use root test

Ex: $\sum \left(\frac{4+5n}{6+2n}\right)^n$.

$$\lim_{n\to\infty}\left[\left(\frac{4+5n}{6+2n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{4+5n}{6+2n}$$

$= 5/2 > 1$, so the series diverges.

INTEGRAL TEST

IF $a_n = f(n)$ and $\int f(x)\,dx$ can be easily computed, use integral test.

$\sum n e^{-n^2}$

$$\int_1^\infty x e^{-x^2}\,dx = \lim_{t\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_1^t$$

$= \frac{e}{2} < \infty$, so $\sum n e^{-n^2}$ converges.