MATH 265 PRACTICE FINAL EXAM

Part I

1. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

   (a) For what values of a and b will the system have infinitely many solutions?

   (b) For what values of a and b will the system be inconsistent?

2. Let

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

   (a) Compute $\det(A)$.

   (b) Under what conditions is the matrix A nonsingular?

   (c) Compute A^{-1}, when it exists.

3. Find a basis for the eigenspace associated to $\lambda = 2$ for the matrix

$$A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (a) Find conditions on a, b, c, d such that the matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $AB = BA$.

   (b) Verify that the set V of all matrices B such that $AB = BA$ is a subspace of the vector space $M_{2\times 2}$, and find a basis for it.

5. Consider the system of differential equations

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

where the matrix

$$A = \begin{bmatrix} -1 & -3 \\ 6 & 8 \end{bmatrix}$$

has an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue $\lambda_1 = 2$, and an eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ with eigenvalue $\lambda_2 = 5$.

(a) Find the general solution.

(b) Find the solution to the initial value problem determined by $x_1(0) = 0$, $x_2(0) = 1$.

6. Suppose that $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation satisfying

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \; L\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -14 \end{bmatrix}, \; L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$$

(a) Determine $L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

(b) Find a matrix A such that $L(\mathbf{v}) = A\mathbf{v}$ for all vectors $\mathbf{v}$ in $\mathbb{R}^3$.

7. Let

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

(a) Find a nonsingular matrix P such that $P^{-1}AP$ is a diagonal.

(b) Compute A^{15}.

8. Using Gram-Schmidt, find an orthogonal basis for subspace S of $\mathbb{R}^3$ consisting of vectors

$$\begin{bmatrix} a+b \\ 2a+b \\ a-b \end{bmatrix}, \text{ where } a, b \text{ are arbitrary numbers}$$

9. (a) Do the following vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix}$$

form an orthogonal basis of $\mathbb{R}^3$?

(b) Write $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, that is find c_1, c_2, c_3 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

10. Prove the following two statements.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal

Part II: (Multiple Choice)

Circle the correct answer.

1. Which of the following sets of vectors is linearly independent?

(a) $[1, 1, 1], [2, 2, 5], [0, 0, 1]$.

(b) $[2, 2, 3], [0, 0, 1], [1, 0, 2], [0, 1, 3]$.

(c) $[2, 2, 3], [0, 0, 0], [0, 0, 1]$.

(d) $[1, 0, 2], [2, 2, 3], [0, 1, 3]$.

(e) $[0, 1, 0], [0, 2, 0]$.

2. Let A be an $n \times n$ nonsingular matrix. Which condition is *not* true?

(a) The system $A\mathbf{v} = 0$, with $\mathbf{v}$ in $\mathbb{R}^n$, has non-trivial solutions.

(b) A is row equivalent to the identity matrix.

(c) A can be written as a product of elementary matrices.

(d) The rank of A is n.

(e) The determinant $\det(A) \neq 0$.

3. Let A be a nonzero $n \times n$ matrix. Determine which of the following matrices are *not* symmetric:

(a) $A + A^T$.

(b) AA^T.

(c) A^TA.

(d) $I + A + A^T$.

(e) $A - A^T$.

4. The eigenvalues of

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

are

(a) $\lambda = 1, 2, 4$.

(b) $\lambda = -4, -1, 2$.

(c) $\lambda = -4, -2, 1$.

(d) $\lambda = 2, 4$.

(e) $\lambda = -1, 2$.

5. The inverse of

$$A = \begin{bmatrix} 2-i & 4i \\ -i & 2+i \end{bmatrix}$$

is

(a) $\begin{bmatrix} 2+i & \frac{i}{4} \\ -i & 2-i \end{bmatrix}$

(b) $\begin{bmatrix} 2+i & -4i \\ i & 2-i \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{2}-i & i \\ -\frac{i}{4} & \frac{1}{2}+i \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2}-\frac{i}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2}-\frac{i}{2} \end{bmatrix}$

(e) nonexistent.

6. Which of the given subsets of the vector space $\mathbb{R}^3$ are subspaces?

(i) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a \geq 0$.

(ii) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a - 2b + c = 0$.

(iii) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a^2 + b^2 + c^2 = 1$.

(a) (i) only.
(b) (ii) only.
(c) (i) and (ii) only.
(d) (iii) only.
(e) all of them.

7. For what values of k are the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix}$ linearly dependent?

(a) $k = -2$.
(b) $k \neq -2$.
(c) $k \neq 2$.
(d) $k = 2$.
(e) All values of k.

8. Let A, B be 2×2 matrices such that $\det(A) = 2$ and $\det(B) = 3$. Then the value of $\det(\frac{1}{5}A^2B^{-1}A^TB^2)$ is

(a) $\frac{24}{5}$.
(b) $\frac{48}{5}$.
(c) $\frac{1}{5}$.
(d) $\frac{6}{25}$.
(e) $\frac{24}{25}$.

9. Given that the inner products $(\mathbf{u}, \mathbf{v}) = -2$, $(\mathbf{u}, \mathbf{u}) = 4$ and $(\mathbf{v}, \mathbf{v}) = 7$. The value of $(3\mathbf{u} - 5\mathbf{v}, \mathbf{u} + \mathbf{v})$ is

(a) 4.
(b) 16.
(c) -19.
(d) 12.
(e) -23.

10. Let A be a 6×5 matrix. Suppose that the null space of A is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The rank of A is

(a) 1.
(b) 2.
(c) 3.
(d) 4.
(e) 5.