MA 341 TEST #1 Oct. 2, 2002

Name *Key*

(20 pts) 1) Use the definition to show that $\lim_{n\to\infty}\left(\frac{n^2+2n+2}{5n^2+4}\right)=\frac{1}{5}$.

Must show that for each $\varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$ s.t. for $n>N(\varepsilon)$

$$\left|\frac{n^2+2n+2}{5n^2+4}-\frac{1}{5}\right|<\varepsilon \quad (0)$$

Now:

$$\left|\frac{n^2+2n+2}{5n^2+4}-\frac{1}{5}\right| = \left|\frac{5n^2+10n+10-5n^2-4}{5(5n^2+4)}\right| = \frac{10n+6}{5(5n^2+4)} < \frac{10n+6}{25n^2}$$

$$= \frac{2}{5n}+\frac{6}{25n^2} < \frac{2}{5n}+\frac{6}{5n} = \frac{8}{5n} \quad (2)$$

By Archimedean property, $\exists N(\varepsilon)$, s.t for $n>N(\varepsilon)$

$$\frac{1}{n}<\frac{5\varepsilon}{8}$$

Sub into (2) gives the result (0).

(15 pts) 2) Find $\sup\{x : x=1-1/n,\ n\in\mathbb{N}\}$. Justify all statements in your argument.

Clearly 1 is an upper bound of $S=\{x : x=1-\frac{1}{n},\ n\in\mathbb{N}\}$

Now let v be a real no, $0<v<1$. We shall show that v is not an upper bound of S, so by def

$1=\sup S$.

By Archimedean property, $\exists n_v\in\mathbb{N}$ s.t.

$$n_v>\frac{1}{1-v}>0$$

Hence $\quad 1-v>\frac{1}{n_v}$ and $1-\frac{1}{n_v}>v$.

(15 pts) 3) Define what is meant by a Cauchy sequence and use your definition to show that $\{x_n\} = \left\{\left(\frac{3}{5}\right)^n\right\}$ is Cauchy.

A sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$, $\exists$ a $K(\varepsilon) \in \mathbb{N}$ s.t. for all $n > K(\varepsilon)$, $m > K(\varepsilon)$ $|x_n - x_m| < \varepsilon$

Here $|x_m - x_n| = \left|\left(\frac{3}{5}\right)^m - \left(\frac{3}{5}\right)^n\right|$ Suppose $m > n$

Then $|x_m - x_n| = \left(\frac{3}{5}\right)^n \left|1 - \left(\frac{3}{5}\right)^{m-n}\right| \leq \left(\frac{3}{5}\right)^n$

Since $\left(\frac{3}{5}\right)^n \to 0$, $\exists$ $K(\varepsilon)$ s.t. for all $n > K(\varepsilon)$, $\left(\frac{3}{5}\right)^n < \varepsilon$

Hence for all $m > n$, $|x_m - x_n| < \varepsilon$ ~~for~~

(15 pts) 4) Let $\lim_{n\to\infty} x_n = -1$. Show that there exists an $N \in \mathbb{N}$ such that for $n > N$, $x_n < -0.9$.

Since $\lim_{n\to\infty} x_n = -1$, $\exists$ $K \in \mathbb{N}$ s.t. for $n > K$

$|x_n + 1| < 0.1$

$-0.1 + 1 < x_n < 0.1 - 1 = -0.9$

(15 pts) 5) Use the monotone convergence theorem to show that the sequence $\{x_n\}$ where $x_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ is convergent and to determine $\lim_{n\to\infty} x_n$. Justify your arguments.

$X_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$

$X_{n+1} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}$ $\Rightarrow X_{n+1} - X_n = \frac{1}{n+3} - \frac{1}{n} < 0$

$\therefore$ Sequence is monotone decreasing and bdd below by zero

$\therefore \{X_n\}$ converges to $\inf\{X_n\}$.

Claim 0 is infemium, I shall show that if $v > 0$ then v cannot be a lower bound.

Now $X_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} > \frac{3}{n+2}$

By Arc Prop $\exists K(v)$ s.t. $\frac{3}{K+2} < V$

Hence $X_K < V$ + so 0 is inf.

By monotone conv. thm $X_K \to \inf\{X_K\} = 0$.

(20 pts) 6) State which results established in the text you are using to find:

(a) $\lim_{n\to\infty} \left(\frac{2}{n}\right)^{\frac{1}{n}} \frac{\cos(n!\alpha)}{n}$ Should read

OMIT

EVERYONE GETS 10 PTS.!

(b) $\lim_{n\to\infty}[(\sqrt{n+1}-\sqrt{n})\sqrt{n+\alpha}] \quad \alpha > 0$

$$(\sqrt{n+1}-\sqrt{n})(\sqrt{n+\alpha} = (\sqrt{(n+1)}-\sqrt{n})\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\left(\sqrt{n+\alpha}\right)$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \sqrt{n+\alpha}$$

$$= \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}} \cdot \frac{\sqrt{1+\alpha/n}}{(\sqrt{1+\frac{1}{n}}+1)}$$

$\frac{\alpha}{n} \to 0$ as $n \to \infty$ $\quad \frac{1}{n} \to 0$ as $n \to \infty$

$1+\frac{\alpha}{n} \to 1$ and $1+\frac{\alpha}{n} \to 1$ as $n \to \infty$

since $x_n \to x \Rightarrow \sqrt{x_n} \to \sqrt{x}$ $\quad (x_n \geq 0)$.

Apply quotient Thm to get

$$\lim_{n\to\infty}\left[(\sqrt{n+1}-\sqrt{n})(\sqrt{n+\alpha}\right] = \frac{1}{2}$$