MA 341 TEST #2 November 18, 2002

Name Key

(10 pts) 1) Let $f : A \to \mathbb{R}$, and let c be a cluster point of A.

(a) Define $\lim_{x \to c} f(x) = L$.

For each $\varepsilon > 0$, $\exists$ a $\delta(\varepsilon) > 0$ such that whenever $x \in A$ and $0 < |x - c| < \delta(\varepsilon)$, then $|f(x) - L| < \varepsilon$

(15 pts) (b) Use the definition to show that

$$\lim_{x \to 1} \frac{x^2 + 2x + 3}{x + 1} = 3$$

Let $\varepsilon > 0$ be given. Consider

$$\left| \frac{x^2 + 2x + 3}{x+1} - 3 \right| = \left| \frac{x^2 + 2x + 3 - 3x - 3}{x+1} \right| = \left| \frac{x^2 - x}{x+1} \right| = \frac{|x|\,|x-1|}{|1+x|}$$

If $|x - 1| < \frac{1}{2}$, then $0 < \frac{1}{2} < x < 3/2$ and

$$\frac{|x|\,|x-1|}{|1+x|} < |x|\,|x-1| < \frac{3}{2}|x-1|$$

Let $\delta = \min\left(\frac{1}{2}, \frac{2}{3}\varepsilon\right)$. Then if $|x-1| < \delta$, we have

$$\left| \frac{x^2 + 2x + 3}{x+1} - 3 \right| < \varepsilon$$

(15 pts) (c) Use the definition to show that if $f(x) > 0$ for x in A and $\lim_{x \to c} f(x) = L$, then $L \geq 0$. (Hint: assume $L < 0$)

Suppose $L<0$. Let $\varepsilon = \frac{|L|}{2}$. Then since $\lim_{x\to c} f(x) = L$, $\exists\, \delta$ such that for $x \in A$ and $0<|x-c|<\delta$, we have

$$|f(x)-L| < \frac{|L|}{2},$$

or

$$L - \frac{|L|}{2} < f(x) < \frac{|L|}{2} + L = -\frac{L}{2} + L = \frac{L}{2} < 0$$

Since $L<0$

This contradicts $f(x)>0$ for all $x \in A$.

(10 pts) 2) (a) Define: "f is uniformly continuous on A." For each $\varepsilon > 0$, $\exists$ a $\delta(\varepsilon)$ such that for all x, u in A with $|x-u|<\delta$,

$$|f(x)-f(u)|<\varepsilon.$$

(15 pts) (b) Use the definition to show that $f(x) = \dfrac{x}{1+x}$ is uniformly continuous on $[0, \infty)$.

$$|f(x)-f(u)| = \left|\frac{x}{1+x} - \frac{u}{1+u}\right| = \frac{|x-u|}{|1+x||1+u|} = \frac{|x-u|}{(1+x)(1+u)} \leq \frac{|x-u|}{1\cdot 1}$$

Since $x \geq 0$, $u \geq 0$

$\therefore$ Given an $\varepsilon>0$, Take $\delta = \varepsilon$.

(10 pts) (c) Show that $f(x) = \dfrac{1}{x}$ is NOT uniformly continuous on $(0, \infty)$.

We must show that $\exists \varepsilon_0 >$ such that for each $k > 0$, $\exists x_k, u_k$ in $(0, \infty)$ such that $|f(x_k) - f(u_k)| > \varepsilon_0$

For each k, take $x_k = \frac{1}{k+1}$ $u_k = \frac{1}{k}$. Then $|x_k - u_k| = \left|\frac{1}{k+1} - \frac{1}{k}\right| = \left|\frac{-1}{k(k+1)}\right|$

$$= \frac{1}{k(k+1)} < \frac{1}{k}$$

and

$|f(x_k) - f(u_k)| = |(k+1) - k| = 1$

$\therefore$ Take $\varepsilon_0 = \frac{1}{2}$

(15 pts) 3) Let f be continuous on the closed bounded interval $[a, b]$. Let $\{x_n\}$ be a sequence of points in $[a, b]$ such that $\lim_{n\to\infty} |f(x_n)| = 0$. Show that there exists a point c in $[a, b]$ such that $f(c) = 0$.

By Bolzano Weierstrass Theorem $\exists$ a subsequence $\{x_{n_k}\}$ and a point c in $[a,b]$ such that $\lim_{k\to\infty} x_{n_k} = c$.

Then: $\lim_{k\to\infty} |f(x_{n_k})| = 0$, and also $\lim_{k\to\infty} f(x_{n_k}) = 0$ (1)

On the other hand, since f is continuous, ~~lim~~ and $\lim_{k\to\infty} x_{n_k} = c$, by the sequential criterion for continuity

$$\lim_{k\to\infty} f(x_{n_k}) = f(c).$$

Comparing this with (1) gives $f(c) = 0$.

(10 pts) 4) Show that for $x > 0$

$$\frac{x}{1+x^2} < \tan^{-1} x < x.$$

(Hint: $\tan^{-1} 0 = 0$ $\quad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$ Mean Value Theorem).

By the Mean Value Theorem

$$\tan^{-1} x = (\tan^{-1} 0) + \left[\frac{d}{dx}(\tan^{-1} x)\Big|_{x=c}\right](x-0)$$

where $0 < c\ x$

$\therefore \quad \tan^{-1} x = \frac{1}{1+c^2} x \qquad (1)$

Since $c > 0 \quad \frac{1}{1+c^2} x < x \qquad (2)$

Since $c < x, \quad \frac{1}{1+c^2} > \frac{1}{1+x^2} \quad (3)$

Combining (1) (2) (3) gives

$$\frac{x}{1+x^2} < \tan^{-1} x < x$$