Solutions to Selected Problems

H.W Lesson 2

6(c)

Let $\varepsilon > 0$ be given. Then

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| = \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \quad (1)$$

By Arch. prop $\exists N(\varepsilon) \in \mathbb{N}$ s.t. $\frac{1}{N(\varepsilon)} < \varepsilon^2$.

Hence for $n > N(\varepsilon)$, $\quad \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon^2$

and so $\frac{1}{\sqrt{n}} < \varepsilon$.

$\therefore$ From (1) for $n > N(\varepsilon)$

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| < \varepsilon.$$

(8) $\lim\limits_{n\to 0} x_n = 0 \iff \lim\limits_{n \to 0} |x_n| = 0$

Pf. $\Rightarrow$

$$||x_n| - 0| = |x_n| = |x_n - 0| \quad (1)$$

$x_n \to 0 \Rightarrow$ For each $\varepsilon > 0$, $\exists N(\varepsilon)$ s.t for $n > N(\varepsilon)$ $|x_n - 0| < \varepsilon$

$\therefore$ for $n > N(\varepsilon)$ $||x_n| - 0| < \varepsilon$

$\Leftarrow$

$|x_n - 0| = ||x_n| - 0|$ & so by argument similar to above

$|x_n| \to 0 \Rightarrow x_n \to 0$

(10) Let $\varepsilon = x$. Since $x_n \to x$, $\exists M$ ~~$\exists N(\varepsilon)$~~ s.t. for $n >$ ~~$N(\varepsilon)$~~ M

$$-x < x_n - x < x$$

$$0 < x - x < x_n < 2x$$

& so for $n > M$, $x_n > 0$.

11) $\frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{1}{(n-2)!} \frac{n}{n-1}$

For $n > 2$, $\frac{n}{n-1} < 2$, & $\frac{1}{(n-2)!} < \frac{1}{n-2}$ for $n > 5$

$\therefore$ For $n > 5$

$$\frac{n^2}{n!} < \frac{1}{(n-2)} \, 2$$

Let $\varepsilon > 0$ be given. By Archimedean prop $\exists N(\varepsilon)$

s.t. for $n > N(\varepsilon)$ $(n-2) > \frac{2}{\varepsilon}$ & so $\frac{1}{(n-2)} < \varepsilon/2$

Hence for $n > N(\varepsilon)$ $\left|\frac{n^2}{n!} - 0\right| < \varepsilon$.